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Mathematical Theory of the Fluctuation of Acoustic Signals in the Ocean (Part I)

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The mathematical theory of the fluctuations of acoustic signals in transit through the ocean is reviewed in detail. First the experimental basis of the theory is presented using recent data. Then a series of mathematical equations are critically reviewed as to their validity and range of application in describing acoustic fluctuations. Then a list of solutions of these equations is discussed, including Born, Parabolic, Dyson, Bethe-Salpeter, Markov, DeWolf, Beran, Mode Coupling, Diffracting Screens, Middleton, Monte Carlo Ray-Optic, Palmer, and DeSanto.
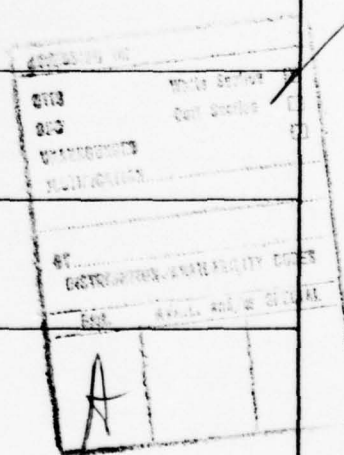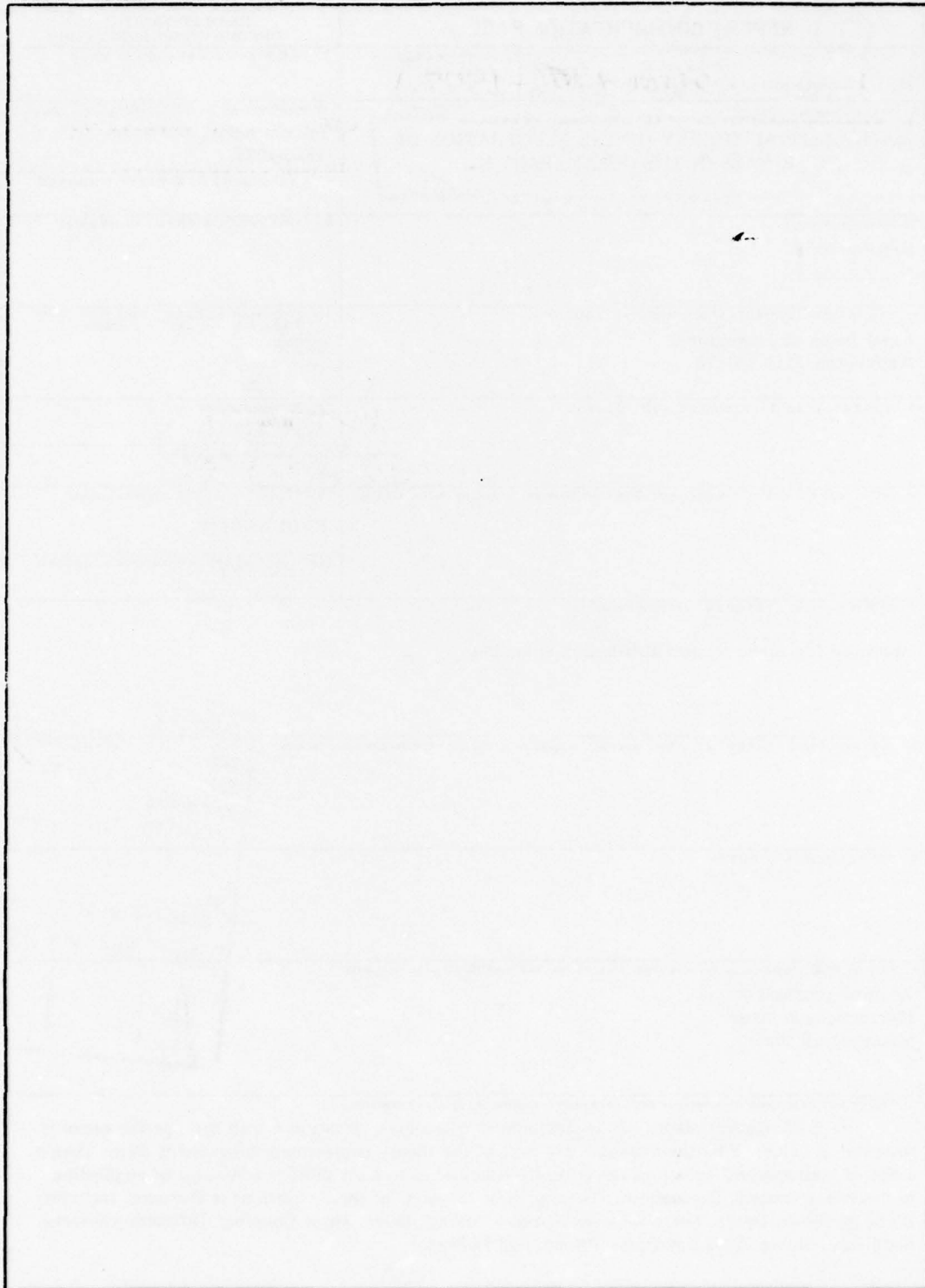
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MATHEMATICAL THEORY OF THE FLUCTUATION OF
ACOUSTIC SIGNALS IN THE OCEAN
(PART I)

Introduction and Statement of the Problem

The propagation of sound fields in inhomogeneous media bounded by impedance boundaries can be represented as propagation in a homogeneous unbounded medium energized by true sources (when present) distributed throughout the medium, and by fictitious sources arising from scattering from volume distributed inhomogeneities, and from impedance boundaries. In this approach the medium discontinuities (volume distributed or surface distributed) are replaced by sources receiving energy from the sound field itself (hence fictitious) rather than energy from an external agency.

The interaction of sound fields with discontinuities in the medium can be regarded as a differential-integral operation between sound and matter. Discontinuities in the medium arise from changes in compressibility, density, momentum (and other causes). Thus we represent the propagation of sound in inhomogeneous media by the (verbal) equation,

D'Alembertian of the sound pressure = true (i.e. external energy) volume sources

(monopole, dipole, quadrupole, etc) + fictitious volume sources (due to changes

in compressibility, density, momentum, etc) + fictitious boundary sources.

Fictitious sources can be mathematically modeled by choosing the interaction of sound and discontinuity as the input-output of a filter, taken in the first approximation to be linear. Both deterministic and stochastic filter representations are needed in the general case. Nonlinear filter representations are needed in special cases.

In particular cases where the scattering of sound rather than its propagation is under study one considers only fictitious sources. The symbolic equation for scattering is then given by

$$\Box^2 p_{scatt}(\underline{x},t) = \mathcal{U}_V(\underline{x},t \mid x_0,t_0) p(\underline{x}_0,t_0)$$
$$+ \mathcal{U}_s(\underline{x},t \mid x_s,t_s) p(\underline{x}_s,t_s)$$

in which $\mathcal{U}_V, \mathcal{U}_s$ are linear stochastic operators (analogous to input-out operations in filters), p is the total sound field and p_{scatt} is the scattered sound field.

Note: Manuscript submitted March 18, 1977

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In the following series of reports the nature the operators $\mathcal{M}_V, \mathcal{M}_S$ is explored in detail. By the use of explicit forms for $\mathcal{M}_V, \mathcal{M}_S$ one arrives at an integral-differential stochastic equation in the unknown stochastic scattered field p_{scatt} . The calculation of the statistical moments of p_{scatt} is the key goal of this study and will be studied in later reports of this same series. In all cases illustrations of method and of results will be taken from the existing literature with emphasis on latest (1976) data and results.

We commence this study with a statement of the experimental basis of fluctuations in the ocean.

1. Experimental Basis of the Theory of Ocean Fluctuations

The partial coherence and fading of underwater acoustic signals resulting from long range transmission have been investigated experimentally in connection with their effect on the performance of underwater receiver arrays. A typical measurements plan to obtain data at sea is discussed below.

Measurements Plan

A CW source of appropriate frequency and fixed amplitude is first selected. The stability of the frequency must be controlled to within very stringent limits in order to assure high source phase accuracy. A precise knowledge of frequency is afforded by a counting circuit which itself is driven by a stable oscillator. A second oscillator circuit is used to provide a reference signal to be used in processing the received signal at some distant point.

The transmission path is next selected to conform to the range under investigation. Often both source and receiver arrays are placed in the sound channel.

The receiver consists of an array of hydrophones in which the spatial disposition and interelement spacing constitutes the key geometric parameters. The received signal in each hydrophone is (individually) demodulated. A conventional but very useful demodulator consists of these items: (1) a band pass filter to improve the signal to noise ratio, with a wide enough bandwidth to admit fluctuations in the frequency of the signal, but still very narrow compared to the CW signal, (2) a dual phase shifting network and demodulator to extract quadrature components of the signal, (3) a low pass filter to exclude high frequency modulator noise, (4) a sampling switch to sample the random demodulated signal. The sampling rate must be carefully adjusted to satisfy the Nyquest criterion, and avoid aliasing.

In summary the measurements plan provides the following data at each hydrophone station: (1) a random received signal in analog form, (2) a demodulated signal in analog form, (3) quadrature components of the received signal in analog form (4) quadrature components in digitized form.

Data Analysis

The signal at the l'th hydrophone requires an appropriate model. A convenient choice is the sinusoidal constructive model. This model is based on the generic form of the emitted signal,

$$Z(t) = Z_c \, e^{-j\omega_0 t} \tag{1.1}$$
$$Z_c = A(t) \exp\left[-j\Phi(t)\right]$$

The changes in the signal induced by the transit through the medium from source to receiver are modeled in the following (possible) ways:

a. Echo signal with constant lag and constant change in time scale. In this model time t in the received signal is delayed by range transit time τ_s, and doppler shift $\nu\omega$, where $\nu = (2\upsilon/c)\cos\theta$, υ is speed of target, c is the speed of the signal, and θ is the angle between target motion and receiver-target line of sight. Thus the received signal Z_s is

$$Z_s = a \, A(t(1+\nu)-\tau_s) \exp\left\{-j\Phi(t(1+\nu)-\tau_s)\right\} \exp\left\{-j\nu\omega_0(t-\tau_s)\right\} \tag{1.2}$$

The symbol a is a constant scale factor.

b. Echo signal with fluctuating amplitude. In this model the only effect of the medium is to randomize the amplitude of the emitted signal.

$$Z_s = A_s \, Z_c(t) \tag{1.3a}$$

in which A_s is a random process. Note that t is "receiver time."

c. Echo signal with a random additive component. In this model the received signal has the same form as the source signal to which is added a random stationary process with

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mean value zero:

$$Z_s = Z_c(t) + X(t), \qquad \langle X(t) \rangle = 0 \qquad \text{(1.3b)}$$

Here t is local receiver time.

    d. Echo signal with <u>random amplitude modulation.</u> The effect of the medium is modeled here as a random amplitude modulation, viz.

$$Z_s = Z_c(t)\left(1 + m_A X(t)\right) \qquad \text{(1.3c)}$$

in which m_A is the effective coefficient of modulation, and $X(t)$ is a stationary random process with mean value zero and variance of unity.

    e. Echo signal with <u>random phase modulation.</u> The form of this model is given by,

$$Z_s(t) = Z_c(t)\, exp\left(-j\,\Psi(t)\right) \qquad \text{(1.3d)}$$

in which $\Psi(t)$ is a random phase increment with zero mean.

    f. Echo signal with <u>random frequency modulation.</u> The mathematical form of this model is the same as that of random phase modulation, excepting that the random phase is due to time fluctuating causes:

$$\Psi(t) = \int_0^t \omega(t')\,dt'$$

$$\omega(t) = \left(\frac{2\omega_0}{c}\right) V(t) \qquad \text{(1.3e)}$$

in which $V(t)$ are random fluctuations in the relative movement of the reflecting object.

    g. <u>Echo signal is a sum of elementary signals.</u> In this model the emitted signal $Z(t) = Z_c(t)\, exp(-j\,\omega_0 t)$ appears at the receiver as a <u>superposition of elementary signals</u> of the same form but each modified by a random amplitude a_k, a random momement of arrival t_k, and a random doppler shift of frequency ω_k. Thus,

$$Z_s(t) = \sum_{k=1}^N a_k Z_c(t-t_k)\, exp\left[j\,\omega_0 t_k - j\,\omega_k(t-t_k)\right] \qquad \text{(1.3f)}$$

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When the model for the signal at the l'th hydrophone is selected it is then necessary to choose an appropriate statistical description of the data based on this model. In most cases the physical event of importance is the wavefront coherence of the signal field, and the signal fading as a function of time. Statistical descriptions of random fields are reviewed in Sects. 9, 10. Statistical analysis can be made on the total received signal and/or on its quadrature components and/or on its phase and amplitude separately.

Statistical averaging may be based on ensemble averaging of many realizations of a random process at specific points in time and space, or upon a single realization in time between several points in space. The single realization procedure is customarily adopted in conventional processing. A key problem associated with it is the need to choose the length of "averaging time" so as to include the fluctuations under study, yet insure stationarity of the data when a stationary model is in question. If non-stationary models are under investigation the averaging time can be extended beyond that of stationary models.

Experiments of Parkins and Fox (IEEE AV-19, 158 (1971))

The range selected was 700 mi between Eleuthera and Bermuda. Both sources and receiver were located in the deep sound channel between these stations. The source frequency was 366.56 Hz and the emitted signal was a sinusoid of constant amplitude and phase. The receiver consisted of several sets of hydrophones distributed over an aperture of 2000 ft. The received signals were band limited to 2 Hz centered at the carrier frequency and sampled at the rate of 9.75 samples/sec. over several 12-h periods from Dec. 1967 to April 1968.

The model selected for the received signal $\psi_\ell(t)$ at the l'th hydrophone was a combination of random amplitude and random phase (choices (b) and (e) above), viz.

$$\psi_\ell(t) = A_\ell(t) \cos\left[\omega_0 t + a_\ell(t)\right] \tag{1.4}$$

The key statistical description in the digital data analysis was the mutual coherence function Γ computed on the basis of one time realization simultaneously recorded at two hydrophone locations (one at ℓ and the second at k). By definition,

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$$\Gamma_{1,2}(x_1, x_2; \tau) \equiv \frac{1}{2T} \int_{-T}^{T} A(x_1, t) A(x_2, t+\tau) \tag{1.5}$$

$$\times \exp\left\{ i \left[a(x_1, t) - a(x_2, t+\tau) \right] \right\} dt$$

The averaging time T was selected to be 105 s for stationary model analysis and 840 s for nonstationary model analysis. The calculation of Γ was expedited by use of FFT techniques.

Two auxiliary functions were then obtained from a knowledge of $\Gamma_{\ell k}$. These were (a) the <u>degree of mutual coherence</u> $\delta(x_1, x_2; \tau)$ given by

$$\delta(x_1, x_2; \tau) \equiv | \delta(x_1, x_2; \tau) | e^{i\theta(x_1, x_2, \tau)} = \frac{\Gamma(x_1, x_2; \tau)}{\Gamma^{1/2}(x_1, x_1; 0) \Gamma^{1/2}(x_2, x_2; 0)} \tag{1.6}$$

in which the level $|\delta|$ and phase θ separately play important roles and (b) the <u>array gain</u> G given by

$$G = \frac{\displaystyle\sum_{i}^{N} \sum_{j}^{N} \Gamma(x_i, x_j; 0)}{\displaystyle\frac{1}{N} \sum_{i=j}^{N} \Gamma(x_i, x_j; 0)} \tag{1.7}$$

Both "steered" and "unsteered" arrays were examined.

A second group of statistical analyses was made on the quadrature components of received signals, and on their amplitudes and phases. These analyses included (a) sample probability density functions (b) sample means, (c) sample variances (d) power spectral densities.

The conclusions drawn by Parkins and Fox from their analyses of the data were as follows:

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(1) A plot of level of coherence $\left| \delta(x_i, y_{\cdot}; \tau) \right|$ versus time for individual hydrophones was <u>oscillatory</u> in the range $0 < \tau < 30$ sec. Thus the field at one hydrophone became incoherent at 5 sec of lag, then recovered partial coherence (at a much lower fraction) at 10 sec. of lag then became incoherent again at 20 sec of lag, etc.

The phase of coherence $\theta(x_i, x_i; \tau)$ versus lag time τ was also oscillatory between ~ 0.4 radian plus and 0.1 radian minus. A "period" of phase oscillation was about 12 sec.

(2) Two hydrophone spaced 900 ft. apart showed a level of coherence (= $\left| \hat{\delta} \right|$) of approximately 0.25, and oscillated within small excursions of about 0.05 units around this value over a lag range of 0 to 30 sec. Similarly the phase of coherence oscillated about the value of 0.5 radian with an excursion of 0.10 radian.

(3) A plot of the level of coherence between two hydrophones at zero time lag for separation distance (in feet) showed a "best-fit" linear drop from a value of unity at zero separation to about 0.1 at 1000 ft. separation, followed by a slow rise to about 0.2 at 2000 ft. The longest averaging times (=840 sec) were used in these estimates. With shorter averaging time (=105 sec) the level of coherence was much higher at 1000 ft (estimated at $\left| \delta \right| = 0.4$). This indicates that less scattering occurs in the shorter averaging time, and points to the validity of the assumption of a <u>frozen ocean</u> if T is small enough (say less than 2 min.).

(4) The phase $\theta(x_i, y_i; \tau)$ of the degree of coherence was measured relative to that of a plane wave and the results plotted versus separation distance between hydrophone locations over a 2000 ft. aperture. The plot shows erratic scatter of phase between $\pi/2$ and $-\pi/2$, indicating distortion of the wave front from the condition of planarity. Although the wavefront is "corrugated," it does not, on the average, deviate from that of a plane wave.

(5) The maximum array gain for seven equally spaced hydrophones on a 150-ft base line is theoretically 16.9 dB. The measured gain was 13 dB with the use of 840-s averaging time, and 12.3 dB for 105-s averaging time. When the base line was increased to 920-ft the measured gain for 840-sec averaging time was 9.8 dB, not much higher than that for the noise field itself, which was 8.5 dB. The shorter averaging time of 105-s caused the array gain to increase to 12 dB.

(6) The acoustic noise field measured at the hydrophones were shown to a high degree to have Gaussian shaped probability density for quadrature components, Rayleigh shaped probability density for amplitude, and uniform distribution for phase. The signal field quadrature components were Gaussian in the majority of cases calculated. The probability distribution for signal amplitudes was, for most cases calculated, not Rayleigh, while a few were. The probability distribution for echo signal phase was non-uniform in some cases, and uniform in others.

(7) With plane-wave steering there appeared a severe degradation in the signal gain of the array, due to distortions of planarity, and reduced levels of coherence over the array.

(8) The power spectral density of one quadrature component was plotted versus frequency over a spread of 0 to 4 Hz. At 0 Hz the contribution for the carrier wave was (at 366.56 Hz) the maximum of the entire curve. With a slight increase in frequency (less than 0.01 Hz) the spectral density dropped radically (up to 20 dB in some cases). At 0.08 Hz (signal at 366.64 Hz) the spectrum rose again to a sharp peak, with a width of about 0.10 Hz. This rise is characteristic of the test situation and indicates interaction of the sound wave with the rough, moving surface. The spectrum then drops linearly about 30 dB in a span of 1 Hz. Beyond this the spectrum remains level at a value of about –35 dB in the frequency range from 1.0 to 3.0 Hz, indicating the presence of ambient sea noise.

2. Experimental Measurements of Acoustic Signals in the Ocean

Qualitative Remarks on the Distinction Between Pulse and CW Transmission

Stanford (JASA 55 968 (1974)) placed a projector at Eleuthera 527 m deep, and beamed a 367 Hz signal at 216 dB/μPa at 1 meter to Bermuda where two point receivers were located at depths 1683 m and 1723 m respectively. Records were made of 1h pulsed signals, 87 msec long, at 10 sec. repetition rate, then 72 h of CW, then 1 hr again of 87 msec. pulses.

A typical received pulsed signal showed high amplitude spikes at 10 sec intervals, most likely arriving at the receiver via a surface reflection, each followed by a train of lower level fluctuating arrivals representing signals from many totally refracted paths. It is observed: A record of received CW signals depends on receiver depth and time. Over several minutes the reception may be stable at one receiver depth, and exhibit large and

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rapid amplitude fluctuations at a second receiver depth. In the next several minutes the roles of the two receivers may be reversed. The projector emits CW energy into many ray paths. If path arrivals at a given time are few in number a chance destructive interference can cause momentary extinction. If a large number of path arrivals exists at a given time the signalfluctuates, and the probability of total extinction is small.

The environmental conditions most directly affecting signal propagation are sea-surface roughness (and motion), and the presence of internal waves along the path.

Amplitude Fluctuation Records (Stanford loc. cita)

The received amplitude data was sorted out into a number of sequential cells of varying magnitude but equal width. From these cells amplitude histograms were constructed of 3 h data. A statistical analysis of the histograms then gave mean level, standard deviation, maximum level and coefficient of variation (= standard deviation/mean). A typical statistical statement on amplitude is this:

Receiver Depth	Month	Mean Level	Max. Level	Coeff. of Variation (%)
1683 m	March	91.8 dB/μPa	100.9	52.7
1723 m	March	91.2	99.7	53.0

Other values are found in Stanford (loc. cit). The ambient noise level was always at least 10 dB less than the average received signal. An average S/N ratio of 20-25 dB was achieved. From the histograms a probability distribution (of amplitude) was constructed. Comparison of the probability density functions by the chi-square goodness test showed that the experimental density function for amplitude distribution were most like modified Gaussian and chi-square functions. One concludes that a sample probability density function for amplitude fluctuations is almost Gaussian.

A set of autocorrelation records of total amplitude fluctuations over a 66 hr. period showed a correlation time (i.e., time required for the normalized autocorrelation to reach a value of 0.1) of 4.8 to 7.3 minutes independent of season.

A Fourier transform of these autocorrelation functions (smoothed by use of Hanning shading) shows three identifiable frequency regions in the resultant power spectrum of amplitude fluctuations. In the frequency range 0.2 to 3 cycles/hour the spectrum level is almost constant. This is the result of interacting internal waves which produce of field of turbulence in the path of the rays, thereby causing amplitude fluctuations in the propagating wave. In the second region (5 to 20 cycles/hr) the spectrum level of amplitude

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fluctuations decreased at a rate approximately $(1/f)^3$ to $(1/f)^4$. This spectrum is closely associated with the spectrum of temperature fluctuations over 3 to 30 cycles/hour which also varies at $(1/f)^3$. The physical causes of these temperature fluctuations is assumed to be internal waves in the immediate vicinity of the main thermocline which modulate temperature and acoustic fluctuations at frequencies less than the Brunt-Vaisala frequency. In the region 20 to 120 cycles/hr the power spectrum of amplitude fluctuations decreases as $(1/f)$. It is assumed to be caused by surface wave roughness, subject to seasonal changes and random winds. Seasonal changes of 9 dB in spectrum level have occurred between March and July.

Phase Fluctuation Records

Phase records are made by comparing the received signal with a reference signal at a controlled frequency. In these (Stanford) tests the reference was a rubidium-vapor frequency standard. Since the phase records were too short to account for the lowest frequencies (0.2 cycle/hr or less) these frequencies were filtered out. A set of 66 hour phase records was then subject to an autocorrelation analysis from which correlation times of 35-46 min. were calculated. A power spectrum density curve for phase, constructed from this autocorrelation, showed a 6 dB per octave negative slope between 0.5 and 100 cycles/ hour, subject at the upper frequency end to season and wind.

Project MIMI

Dyson, Munk and Zetler (JASA 59, 1121 (1976)) have presented a theory of multi-path interference to account for the observed rate-of-phase and intensity spectra of project MIMI. A brief review of their model is given in Section 11.2 of this report. The observed data had the following character: a CW signal at 406 Hz was emitted from a deep water source at Eleuthera and propagated through the deep round channel to a hydrophone receiver at Midstation, 550 km away, and then continued on its way to a second receiver at Bermuda, 1250 km away. The record length of the received signal at Midstation was 8255 five-minute segments (approximately 29 days); at Bermuda it was 7366 five-minute segments (approximately 25.8 days). The received acoustic data was plotted as a time-record of intensity I indecibels and phase Φ in cycles at the rate of 12 readings per hour, each reading being the average of a five minute record. From these I and Φ plots the Cartesian components $X(t)$, $Y(t)$ of the acoustic pressure amplitude $R(t)$ were

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obtained by setting $X(t) = 10^{I(t)/20} \cos 2\pi \Phi(t)$, $Y(t) = 10^{I(t)/20} \sin 2\pi \Phi(t)$, with $I = 20 \log_{10} R$. Since all acoustic quantities were random it was convenient to write each as a sum of a mean value and a fluctuating value: $X = \langle X \rangle + \delta X$, $Y = \langle Y \rangle + \delta Y$, $I = \langle I \rangle + \delta I$, $\Phi = \langle \Phi \rangle + \delta \Phi$, with $R^2 = X^2 + Y^2$. Examination shows that the time records of the cartesian components X, Y exhibit <u>multipath fading</u>. A fade F is expressible in decibels by choosing a fraction ϵ less than unity and calculating $F = 20 \log_{10} \epsilon^{-1}$. Thus a 20 dB fade means a fraction of $\epsilon = 1/10$ is chosen relative to the intensity reference $I_0 = 10 \log_{10} \langle R^2 \rangle$. In order to study fade-out statistics it proved useful to censor the time records of I vs t by choosing 10 dB, 20 dB, and ∞ dB intensity <u>floors</u> below the reference intensity I_0, and replacing any value of I(t) less than $I_0 - F$ by the value of $I_0 - F$ itself. Thus the intensity and phase statistics were calculated for three types of time record of I(t) and $\Phi(t)$, namely (1) the uncensored record (2) fade-outs greater than 20 dB relative to I_0, replaced by the 20 dB level (3) fade-outs greater than 10 dB, replaced by the 10 dB level. The intensity and phase statistics of the data records, censored in the manner described above (i.e. three headings), are reproduced below from the paper of Dyson et al.

	Midstation			Bermuda		
Range from Eleuthera (km)	550 (nominal)			1250		
Record length	8255 terms = 29.0 days			7366 terms = 25.8 days		
$\langle \Phi^2 \rangle$, $\langle (\delta \Phi)^2 \rangle$ (cycles^2)	548,	0.029		110,	0.060	
F (dB)	∞	20	10		20	10
Number of terms replaced	0	114	1076	0	79	746
$\langle X^2 \rangle$ ($\times 10^{-13}$)	8.73	8.73	8.79	0.668	0.667	0.671
$\langle Y^2 \rangle$ ($\times 10^{-13}$)	8.70	8.70	8.76	0.655	0.655	0.659
$\langle (\delta X)^2 \rangle$ ($\times 10^{-13}$)	6.08	6.08	6.14	0.797	0.797	0.806
$\langle (\delta Y)^2 \rangle$ ($\times 10^{-13}$)	5.92	5.92	5.98	0.790	0.791	0.799
$\langle R^2 \rangle$ ($\times 10^{-13}$)	17.43	17.43	17.55	1.323	1.322	1.330
I_0 (dB)	142.41	...	...	131.22	...	...
$\langle I \rangle$ (dB)	139.22	139.27	139.82	128.61	128.65	129.06
$\langle I^2 \rangle - \langle I \rangle^2$ (dB^2)	38.79	35.87	23.37	31.84	30.18	22.52
$\langle (\delta I)^2 \rangle$ (dB^2)	24.10	21.77	13.09	48.17	45.31	30.34
$\langle \lvert \delta I \cdot \delta \Phi \rvert \rangle$ (dB cycles)	0.55	0.52	0.38	1.15	1.13	0.92

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The entries are in arbitrary pressure units. To correct to absolute level in dB re 1μbar each pressure amplitude in the table (actually derived from I) expressed in dB is diminished by 169.0 dB for Midstation data and 173.0 dB for Bermuda data.

The physical significance of this data is discussed briefly in connection with the Dyson multipath model in Section 11.2.

Canonical Models of Sound Speed Profiles

The chief environmental parameter in the theory of the propagation of sound in the ocean is the sound speed $C(z)$. Experience has shown that one can construct a "typical" sound speed profile $C(z)$ vs z with these characteristics: a minimum value C_1 at some depth z_1, and increasing by a few percent toward the top and bottom of the ocean. The fractional gradient $C^{-1}\partial c/\partial z$ of sound speed is thus a key parameter. Munk (JASA 55, 221 (1974)) has modeled this gradient in terms of the Brunt-Väisälä frequency $N(z)$ (see Sect. IV), namely,

$$C^{-1}\frac{\partial c}{\partial z} = -\frac{\mu}{g} N^2(z) + \gamma_A$$

with

$$\mu = 24.5\, \delta, \qquad \delta = \frac{1+\ell Tu}{1-\ell Tu}, \qquad \ell = 0.049 \ (Avg.)$$

$$\gamma_A = 1.14 \times 10^{-2}\ km^{-1} \ (Avg.)$$

Here z is positive downward (i.e. $z = 0$ is the ocean surface), δ is a parameter associated with salinity, Tu is the "Turner number" which shows the effect of salinity S and temperature T on the stability of the water column, and is defined experimentally as,

$$Tu = 6.15 \left(\frac{\partial S}{\partial z}\right)\!\!\left(\partial T/\partial z\right)$$

For fresh water $Tu = 0$; in the North Pacific intermediate waters Tu = -0.3 and off Bermuda at shallow depths Tu = 0.8. Thus local measurements are necessary to calculate Tu and hence calculate the velocity gradient. On the channel axis $z = z_1$ (where $\partial C/\partial z = 0$) one therefore has

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$$N_1 = \dot{N}(z_1) = 2.13 \times 10^{-3} \delta^{-\frac{1}{2}} \qquad (\sec^{-1})$$

This also requires a local measurement of δ. One can however construct a general model for $N(z)$ which will be typical enough for many calculations. Munk recommends the exponential model. written on terms of a stratification scale **B** valid beneath the thermocline:

$$N(z) = N_0 \, e^{-z/B}$$

Here No and B are experience factors, chosen by Munk to be No = 5.24×10^{-3} sec^{-1} and B = 1.3 km. On this basis the depth of the channel axis is modeled to be,

$$z_1 = B \ln\left(\frac{N_0}{N_1}\right) = 1.16 \text{ km} - \frac{1}{2} B \ln \delta$$

From typically observed values, he takes z_1 = 1.3 km (on average), so that δ = 1.32, and $\mu \approx 32$. Thus

$$N_1 \approx 1.9 \times 10^{-3} \quad \sec^{-1}$$

With these values one can further construct a "canonical sound speed profile". Munk suggests the following,

$$C = C_1 \left[1 + \epsilon (\eta + e^{-\eta} - 1) \right],$$
$$\epsilon = 7.4 \times 10^{-3} \quad (AV.)$$
$$\eta = 2(z - z_1)/B = \frac{1}{0.65}(z - 1.3)$$
$$C_1 = 1.492 \text{ km/sec} \quad (AV.)$$

Now in the presence of an internal wave field with vertical displacements ζ the fluctuation in velocity profile at a fixed depth z is

$$\delta c / c \approx \alpha \, \delta T + \beta \, \delta S + \gamma \, \delta P$$
$$\delta T = \zeta \, \partial_z T_+ , \quad \delta S = \zeta \, \partial_z S , \quad \delta P = \rho g \zeta (\Delta \rho / \rho)$$

Where T, S, P are temperature, salinity and hydrostatic pressure respectively and α, β, γ are experimental. The effect of pressure (δ P) can be shown to be negligible. Using the above relations one arrives at the more convenient formula

$$\delta c / c = N^2 (z) \frac{\mu}{g} \zeta$$

According to the Garrett-Munk model (J. Geophys. Res. 80, 291-297(1975)) the rms value of internal wave displacement ζ is,

$$\zeta_{RMS} = \zeta_{o(RMS)} \left(N / N_o \right)^{-\frac{1}{2}} , \quad \zeta_{o(RMS)} = 7.3 \, m$$

in which ζ_o , No are (extrapolated) surface values. Thus,

$$\left(\frac{\delta c}{c} \right)_{RMS} = \frac{\mu}{g} N_o^2 \, \zeta_{o(RMS)} \left(N / N_o \right)^{3/2}$$

Typical values of rms $\delta c / c$ can be obtained from the exponential model of N. Choosing $\delta = 1$, $\mu = 24.5$ Munk and Zachariasen (Stanford Res. Inst. JSS-75-1) arrive at the following table,

Location	Depth (km)	N (sec^{-1})	rms ζ (m)	rms $\delta c / c$
theromcline	z = 0	5.2×10^{-3}	7.3	4.9×10^{-4}
sound axis	z = 1	1.9×10^{-3}	12.0	1.1×10^{-4}
bottom	z = 4.5	1.7×10^{-4}	41.2	2.8×10^{-6}

These numbers are useful in making reasonable models of sound speed profile and their fluctuations (due to internal waves).

Conclusion of Sects. 1 and 2

The experimental results of Parkins and Fox, Stanford, and MIMI constitute only a small portion of available data of ocean fluctuations. They convey however a good summary of what is to be expected by experimentalists who examine different oceans at different seasons in the frequency range 300-400 Hz. In subsequent reports of this series (of which this report is no. 1) a survey will be made of the attempts to match experimentally determined fluctuations in the real ocean with mathematical models based on the theory of fluctuations. For the present we will continue with a review of basic equations governing the propagation of waves (gravity, acoustic, internal, tidal, etc.) in the ocean.

3. Mathematical Modeling of Ocean Fluctuations and Acoustic Propagation

The partial coherence and fading characteristics of acoustic signals in the ocean require a mathematical model which will organize the experimental data into a useful computational algorithm to allow prediction of propagation and scattering in the ocean. This modeling is difficult because of several factors: (a) the number of parameters appearing in the equations is large (b) the inhomogeneities of the ocean (including surface and bottom) are statistically distributed (c) the statistics are not stationary or homogeneous in the general case (d) the mathematical formulations appear as integral-differential equations whose solutions are (in most cases of interest) not available (e) approximate solutions are all of limited accuracy with regard to distance of propagation into the medium and duration of signal (f) multiple scattering complicates all computational procedures to the point where it is not feasible to make calculations because of cost (g) the data on ocean parameters is known accurately only over limited geographic areas and at selected seasons.

In view of the magnitude of the problem of mathematical modeling of ocean fluctuations it seems appropriate to review the basic formulations of a mathematical model of ocean hydrodynamic processes to the extent required to model acoustic propagation, and then to proceed to state in detail the mathematical models currently in use to explain fluctuations.

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4. <u>Mathematical Models of Sources of Ocean Inhomogeneities I:</u>

(Ref. [1] W. Krauss "Dynamics of the Homogeneous and the Quasihomogeneous Ocean" 1973)

The field functions which describe the hydrodynamics of the ocean are seven in number, viz. three components of velocity $\underline{v} = (u, v, w)$, the pressure p, density ϱ, salinity S and temperature T. There are then seven governing equations.

(1) Conservation of Mass:
$$\partial \varrho / \partial t + \underline{\nabla} \cdot \varrho \underline{v} = 0 \tag{4.1}$$

(2) Diffusion of Salinity:
$$\partial S / \partial t + \underline{v} \cdot \underline{\nabla} S = k_D \nabla^2 S \tag{4.2}$$

(3-5) Equation of motion:

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} + 2 \underline{\Omega} \times \underline{v} = -\frac{\nabla p}{\varrho} - \underline{\nabla} \Phi - \nabla_h \Phi_G + \frac{\nu}{3} \underline{\nabla} \, \underline{\nabla} \cdot \underline{v} \tag{4.3}$$

$\underline{\Omega}$ = angular velocity vector of the earth; Φ = gravitational potential

Φ_G = tidal potential; ν = shear viscosity

(6) Equation of State:
$$f(p, \varrho, T) = 0 \tag{4.4}$$

(7) Conservation of Energy (3 Formulations):

$$(a) \quad \int_V \frac{\partial}{\partial t} \left\{ \varrho \left(\frac{v^2}{2} + \epsilon \right) \right\} dV = - \oint \varrho \left(\frac{v^2}{2} + \epsilon \right) \underline{v} \cdot d\underline{a} + \int_V \underline{v} \cdot \underline{f} \, dV$$

$$+ \oint (\underline{v} \cdot \underline{\underline{t}}) \cdot d\underline{a} - \oint \underline{q} \cdot d\underline{a} \tag{4.5}$$

(ϵ is internal energy per unit volume; $\underline{f}$ is a body force; $\underline{\underline{\tau}}$ is the surface stress tensor; $\underline{q}$ is the heat flux vector).

(b) or, First Law of Thermodynamics:

$$\frac{\partial Q}{\partial t} = \frac{1}{\varrho} \left(\underline{\nabla}\underline{v} : \underline{\underline{t}}_F - \underline{\nabla} \cdot \underline{q} \right) \tag{4.6}$$

($\underline{\underline{t}}_F$ is the friction tensor).

(c) or Equation of Heat Conduction: $\quad \partial T / \partial t = k_T \, \nabla^2 T$ \hfill (4.7)

In mixing processes of the real ocean the molecular transport terms of these seven equations are negligible, i.e., $\underline{v} \cdot \nabla \varrho$, $\underline{v} \cdot \nabla S$, $\underline{v} \cdot \nabla \underline{v}$, and $\underline{v} \cdot \nabla T$ can be neglected.

Several models of the ocean based on reduced versions of the basic equations are in current use. These are:

A. The Incompressible, homogeneous and barotropic ocean.

Incompressible: $\left(\partial \varrho / \partial p \right)_{S,T} = 0$ \hfill (4.8)

homogeneous: $\quad \varrho = \varrho_0 = \text{const.} \; ; \; T = T_0 = \text{const.} \; ; \; S = S_0 = \text{const.}$ \hfill (4.9)

barotropic: equi-scalar surfaces (say, of density) coincide with isobaric surfaces.

The equations of this model are,

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} + 2\, \underline{\Omega} \times \underline{v} = -\frac{\nabla p}{\varrho_0} - \nabla \Phi - \nabla_h \Phi_G + \nu \nabla^2 \underline{v} \tag{4.10}$$

$$\nabla \cdot \underline{v} = 0 \, , \quad \nabla_h = \hat{\imath}\, \partial/\partial x + \hat{\jmath}\, \partial/\partial y .$$

Any liquid element in this model placed anywhere is in equilibrium.

B. Incompressible inhomogeneous ocean without mixing.

Inhomogeneous: ϱ, T, S are functions of location.

The equation of this model are,

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} + 2\, \underline{\Omega} \times \underline{v} = -\frac{1}{\varrho} \nabla p - \nabla \Phi - \nabla_h \Phi_G + \nu \nabla^2 \underline{v} \tag{4.11}$$

$$\underline{\nabla} \cdot \underline{u} = 0 \tag{4.12}$$

$$d\rho/dt = 0 \tag{4.13}$$

A liquid element in this model when removed from one location to another is subject to a restoring force, i.e., such an ocean can sustain ocillations in the form of internal waves.

C. Boundary Conditions.

(1) A surface of discontinuity $F(\underline{x}, t) = 0$ moving through a volume V with velocity $\underline{C}$ divides V into V_I, V_{II}, having surfaces σ_I, σ_{II} respectively. For a scalar fluid variable ψ, the conservation law across F requires that

$$\left(\varrho \psi \underline{V} \right)_I - \left(\varrho \psi \underline{V} \right)_{II} - \left[(\varrho \psi)_I - (\varrho \psi)_{II} \right] \underline{C} + \left(\underline{q}_{\psi_I} - \underline{q}_{\psi_{II}} \right) \cdot \underline{n} = 0 \tag{4.14}$$

in which $\underline{V}_I, \underline{V}_{II}$ are the fluid velocities at surfaces σ_I, σ_{II}, and $\underline{q}_I, \underline{q}_{II}$ are source densities of ψ.

(2) When ψ represents mass per unit mass, then $\psi = 1$. In this case for oceanic boundaries, the normal components of $\underline{V}_I, \underline{V}_{II}$ and $\underline{C}$ are identical. This is the kinematic boundary condition. For such a condition, the basic equation at the boundary is

$$\left(\underline{q}_{\psi_I} - \underline{q}_{\psi_{II}} \right) \cdot \underline{n} = 0 \tag{4.15}$$

Also, the total rate of change of surface F is

$$\frac{\partial F}{\partial t} + \underline{V} \cdot \underline{\nabla} F = 0 ; \quad \text{or} \quad \left(\underline{V}_I - \underline{V}_{II} \right) \cdot \underline{\nabla} F = 0$$

If F is the surface of the sea then $F(\underline{x}, t) = z + \zeta(x, y, t) = 0$

Then the z- component of velocity (=W) is $z = -\zeta(x, y, t)$ is,

$$w = -\left(\frac{\partial \zeta}{\partial t} + \underline{V}_h \cdot \underline{\nabla} \zeta \right), \qquad \underline{V}_h = \text{mean horizontal surface velocity}$$

$$\tag{4.16}$$

i.e. the velocity has both local and convective components.

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(3) When ψ represents momentum transport per unit mass then the dynamic boundary condition $\left(\underline{q}_{\psi_{\text{I}}} - \underline{q}_{\psi_{\text{II}}} \right) \cdot \underline{n} = 0$ reduces to

$$\left(\underline{\underline{t}}_{\text{I}} - \underline{\underline{t}}_{\text{II}} \right) \cdot \underline{n} = 0 \quad ; \quad \underline{\underline{t}} = \hat{e}_i \, t_{ij} \, \hat{e}_j \tag{4.17}$$

or

$$\left[\left(t_{ij} \right)_{\text{I}} - \left(t_{ij} \right)_{\text{II}} \right] \hat{e}_i \cos \left(\hat{e}_j, \underline{n} \right) = 0 \tag{4.18}$$

in which $\underline{\underline{t}}$ is the stress tensor, the components of $\left(\hat{e}_j, \underline{n} \right)$ are α_i, and $\cos \alpha_i$ are the direction cosines given by,

$$\cos \alpha_i = \frac{\partial F / \partial x_i}{\sqrt{\left(\frac{\partial F}{\partial x} \right)^2 + \left(\frac{\partial F}{\partial y} \right)^2 + \left(\frac{\partial F}{\partial z} \right)^2}} \tag{4.19}$$

D. Hydrodynamic Perturbation Models

The basic hydrodynamic equations in a scalar ψ are solvable in specific cases, the exact solutions being $\psi^{(0)}$. Whenever not exactly solvable, the basic equations are approximately solved by adding perturbations $\epsilon^n \psi^{(m)}$ to a known exact solution, the parameter ϵ^m being a measure of the strength of the perturbation. As example, let $\underline{K}$ be the vector force externally applied to a volume element and let $\underline{\omega}$ be a reduced angular velocity vector of the earth with components $(0, 0, -2 \Omega \sin \phi)$, where ϕ is the latitude angle. For this case the Coriolis force is given by $-\underline{\omega} \times \varrho \underline{V}$. Then the basic hydrodynamic equations are simplified to the set

$$\varrho \frac{d\underline{V}}{dt} + \underline{\omega} \times \varrho \underline{V} + \underline{\nabla} p + \varrho \underline{\nabla} \Phi - \mu \nabla^2 \underline{V} = \underline{K} \tag{4.20}$$

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$$d\varsigma/dt + \varsigma \underline{\nabla} \cdot \underline{V} = 0 \qquad (4.21)$$

$$\frac{dp}{dt} - c^2\frac{d\varsigma}{dt} = \varsigma c^2 \beta \frac{dQ}{dt} ; \qquad \beta = \frac{\text{coeffic. of thermal expansion}}{\text{specific heat at const. pres.}} \qquad (4.22)$$

Now assume,

$$\underline{V} = \underline{V}^{(0)} + \underline{V}^{(1)} + \underline{V}^{(2)}$$

$$\varsigma = \varsigma^{(0)} + \varsigma^{(1)} + \varsigma^{(2)} \qquad (4.23)$$

$$p = p^{(0)} + p^{(1)} + p^{(2)}$$

$$Q = Q^{(1)} + Q^{(2)}$$

The first order equations are then

$$\varsigma^{(0)}\frac{\partial \underline{V}}{\partial t} + \underline{\omega} \times \varsigma^{(0)}\underline{V}^{(1)} + \underline{\nabla}p^{(1)} + \varsigma^{(1)}\underline{\nabla}\Phi - \mu \nabla^2\underline{V}^{(1)} = \underline{K}^{(1)} \qquad (4.24)$$

$$\partial\varsigma^{(1)}/\partial t + \varsigma^{(0)}\underline{\nabla}\cdot\underline{V} + W^{(1)}d\varsigma^{(0)}/\partial z = 0 \qquad (4.25)$$

$$\partial p^{(1)}/\partial t - c^2 \partial\varsigma^{(1)}/\partial t + W^{(1)}\left[\frac{\partial p^{(0)}}{\partial z} - c^2\frac{d\varsigma^{0}}{\partial z}\right]$$

$$= c^2\beta\varsigma^{(0)}\frac{\partial Q}{\partial t} \qquad (4.26)$$

For an incompressible medium, with no diffusion or heat input these simplify again to the form

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$$\varsigma^{(o)} \frac{\partial \underline{V}}{\partial t} + \underline{\omega} \times \varsigma^{(o)} \underline{V}^{(l)} + \nabla p^{(l)} + \varsigma^{(l)} \nabla \underline{\Phi} = K^{(l)} \qquad (4.27)$$

$$\partial \varsigma^{(l)} / \partial t + w^{(l)} \partial \varsigma^{(o)} / \partial z = 0 \qquad (4.28)$$

$$\underline{\nabla} \cdot \underline{V}^{(l)} = 0 \quad ; \quad \varsigma^{(o)} = \varsigma^{(o)}(z) \qquad (4.29)$$

The vector velocity $\underline{V} = (u, v, w)$. It is desired to find an equation in the vertical velocity component w. This is accomplished by the following steps: (1) the horizontal divergence ($\nabla_{\perp} \equiv (\partial/\partial x) \hat{\imath} + (\partial/\partial y) \hat{\jmath}$) of Eq. (4.27) is taken. Noting that

$$\nabla_{\perp} \varsigma^{(l)} \cdot \underline{\nabla} \underline{\Phi} = 0, \quad \nabla_{\perp}^2 \underline{\Phi} = 0, \quad \underline{\nabla}_{\perp} (\underline{\omega} \times \underline{V}) = -\underline{\omega} \cdot (\underline{\nabla} \times \underline{V})$$

one arrives at

$$\varsigma^{(o)} \underline{\nabla}_{\perp} \cdot \frac{\partial \underline{V}^{(l)}}{\partial t} - \underline{\omega} \cdot (\underline{\nabla} \times \underline{V}^{(l)}) \varsigma^{(o)} + \nabla_{\perp}^2 p + \nabla_{\perp} \cdot K^{(l)} = 0 \qquad (4.30a)$$

(2) Since,

(a)
$$\varsigma^{(o)} \nabla_{\perp} \cdot \frac{\partial \underline{V}^{(l)}}{\partial t} = -\varsigma^{(o)} \frac{\partial^2 w}{\partial z \partial t} \qquad (4.30b)$$

(b)
$$\underline{\omega} \cdot \left(\underline{\nabla} \times \frac{\partial \underline{V}}{\partial t} \right) \varsigma^{(o)} = \omega^2 \varsigma^{(o)} \frac{\partial w}{\partial z} - \underline{\omega} \left(\underline{\nabla} \times K^{(l)} \right)_z \qquad (4.30c)$$

the next step is to differentiate Eq. (4.30a) with respect to time, then substitute (4.30b) and (4.30c), then differentiate the result explicitly with respect to z . (3) The result contains a term $\partial^2 p / \partial z \, \partial t$. This is eliminated by use of the vertical component of the equation of motion (viz. $\partial p^{(l)} / \partial z = g \varsigma^{(o)} - K_z^{(l)} - \varsigma^{(o)} \partial w^{(l)} / \partial t$). The final result is

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$$\nabla_\perp^2 \frac{\partial^2 w}{\partial t^2} + \frac{g}{c} \Gamma \nabla_\perp^2 w^{(1)} + \frac{\partial^4 w^{(1)}}{\partial z^2 \partial t^2} + \omega^2 \frac{\partial^2 w^{(1)}}{\partial z^2} + \Gamma \left(\frac{\partial^3 w^{(1)}}{\partial z \partial t^3} + \omega^2 \frac{\partial w^{(1)}}{\partial z} \right)$$

$$= -\frac{1}{\rho_0} \left[\nabla_\perp \frac{\partial K_z^{(1)}}{\partial t} - \omega \left(\nabla \times \frac{\partial K^{(1)}}{\partial z} \right)_z - \nabla_\perp (\partial^2 K^{(1)} / \partial z \partial t) \right] \qquad (4.31)$$

where

$$\Gamma = \frac{1}{\rho_0} \frac{\partial \rho^{(0)}}{\partial z} \qquad (4.32)$$

If $\partial / \partial t \cdot \frac{1}{c}$, $\underset{\sim}{\omega}$ and $\quad$ K are set to zero, then the first order equation in the vertical velocity is

$$\frac{\partial^4 w^{(1)}}{\partial x^2 \partial t^2} + \frac{\partial^4 w}{\partial z^2 \partial t^2} + g \Gamma \frac{\partial^2 w}{\partial x^2} + \Gamma \frac{\partial^3 w}{\partial z \partial t^2} = 0 \qquad (4.33)$$

Solutions of the Equations of Motion

    In the absence of externally applied forces (i.e. $K = 0$), one first seeks for time-periodic solutions of Eq. (4.31) by setting $w = \overset{0}{w}(\underline{x}) \exp(i\omega t)$. This leads to the following homogeneous wave equations for time harmonic waves,

$$\nabla_\perp^2 \overset{0}{w} - \left[\frac{\omega^2 - (\underset{\sim}{\omega} \cdot \underset{\sim}{\omega})}{g \Gamma - \omega^2} \right] \left(\frac{\partial^2 \overset{0}{w}}{\partial z^2} + \Gamma \frac{\partial \overset{0}{w}}{\partial z} \right) = 0 \qquad (4.34)$$

The algebraic sign of the quantity $\alpha = \left(\omega^2 - \underset{\sim}{\omega} \cdot \underset{\sim}{\omega} \right) / \left(g \Gamma - \omega^2 \right)$ determines the nature of the solutions to this equation. If $\alpha > 0$ then the differential equation is hyperbolic and provides a solution in internal waves.

    Of all these harmonic solutions $\overset{0}{w}(\underline{x})$ one next seeks a further subset consisting of the separated forms,

$$\overset{0}{w}(\underline{x}) = W(z) F(x, y) \qquad (4.35)$$

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Separation leads to two equations in the separation constant a^2, viz.

$$\nabla_\perp^2 F + \left[\omega^2 - \underset{\sim}{\omega}\cdot\underset{\sim}{\omega}\right] a^2 F = 0 \tag{4.36}$$

$$\frac{d^2 W}{dz^2} + \Gamma\frac{dW}{dz} + \left(g\,\Gamma - \omega^2\right)a^2 W = 0 \tag{4.37}$$

The boundary conditions are:

(atmospheric pressure = const.: $\quad \dfrac{dW}{dz} + g\,a^2 W = 0, \qquad z = 0$ $\tag{4.38}$

velocity = 0: $\qquad W = 0, \qquad$ at z = H = bottom. $\tag{4.39}$

A mathematically more convenient form for Eq. (4.37) is

$$\frac{d}{dz}\left[\rho^0(z)\frac{dW}{dz}\right] + a^2 q(z)\,W = 0 \tag{4.40}$$

$$q(z) = \left[N^2(z) - \omega^2\right]\rho^0(z), \qquad N^2(z) = g\,\Gamma \qquad \text{(see Eq. 32)}$$

The three equations (4.37) or 4.40), (4.38), (4.39), constitute an eigen value problem. When $a^2 q(z) > 0$ there are solutions $W(z)$ for specific separation constants which satisfy the boundary conditions (see Eq. (4.44) below). In a physical sense the internal waves are just those <u>vertical</u> motions of the ocean which result from a vertical displacement of water of different density from the surrounding water and which fit the requirement of constant atmospheric pressure at the ocean surface and zero velocity at the ocean bottom, and propagate as progressive waves at harmonic frequency (ω).

To determine a^2 it is noted that the operator $\nabla_\perp^2$ acting upon the field F satisfies the Helmholtz (differential) equation, $\nabla_\perp^2 F = -\varkappa_h^2 F$. Substitution of this relation into Eq. (4.36) gives

$$a^2 = \varkappa_h^2 / (\omega^2 - (\underset{\sim}{\omega}\cdot\underset{\sim}{\omega})) \tag{4.41}$$

Thus,

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$$a^2 \, \mathfrak{z}(\tilde{z}) = \frac{\chi_h^2 \left[N^2(\tilde{z}) - \omega^2 \right]}{\omega^2 - (\underset{\sim}{\omega} \cdot \underset{\sim}{\omega})} \, \varrho^{(0)}(\tilde{z}) \tag{4.42}$$

The essential requirement for the existence of (transverse) internal waves is $a^2 \mathfrak{z}(\tilde{z}) > 0$
or,

$$\frac{N^2(\tilde{z}) - \omega^2}{\omega^2 - (\underset{\sim}{\omega} \times \underset{\sim}{\omega})} > 0 \tag{4.43}$$

It is first supposed that the frequency ω of propagating internal waves is greater than the Coriolis parameter $|\underset{\sim}{\omega}|$. Then the requirement for propagation is $\omega < N(\tilde{z})$, i.e. the period $T (= 2\pi/\omega)$ of the internal waves must be greater than $2\pi / N(\tilde{z})$ which is the shortest scale of time of oscillation, or shortest possible period of internal wave motion. Let N_{MIN} be the minimum Vaisala frequency of the entire water depth H. The periods of all possible internal waves for the entire channel is $T > 2\pi / N_{MIN}$. If the condition $N(\tilde{z}) > \omega$ exists only in the layer $h_1 \leq \tilde{z} \leq h_2$ then for fixed $\omega_0 < N(\tilde{z})$ there are an infinite number of <u>spatially</u> distributed modes of vertical velocity $W_m(\tilde{z})$ $m = 1, 2, \cdots \infty$, inside this layer, which can propagate in the x-direction at phase velocities a_m^{-1}, where a_m satisfies the boundary conditions,

$$(1) \ \frac{dW_m}{d\tilde{z}} + g \, a_m^2 \, W_m = 0 \ , \quad \tilde{z} = 0 \ ; \ (2) \, W_m = 0 \ , \quad \tilde{z} = H$$

with

$$a_m^2 = \chi_{h,m}^2 \Big/ (\omega_0^2 - (\underset{\sim}{\omega} \cdot \underset{\sim}{\omega})), \qquad \omega_0^2 > (\underset{\sim}{\omega} \cdot \underset{\sim}{\omega}) \tag{4.44}$$

or (alternatively)

$$a_m^2 \, \omega_0^2 = \chi_{h,m}^2 + a_m^2 \, (\underset{\sim}{\omega} \cdot \underset{\sim}{\omega}) \equiv K_m^2$$

At frequency ω_0 the wavelengths of these internal waves are given by $\lambda_m = 2\pi/K_m$. and the periods are $\tau^2 = a_m^2 \lambda_m^2$. Outside the layer $h_1 \leq z \leq h_2$ the amplitudes W_m decrease exponentially to their boundary values.

The wave shape in the xy plane is given by $F(x,y)$, which is separable into $G(x)\tilde{G}(y)$ with b^2, d^2 the separation constants in x and y respectively, related to α_h^2 by $\alpha_h^2 - b^2 - d^2 = 0$. Assuming $\alpha_h^2 > 0$ and $\omega > (\underline{\omega} \cdot \underline{\omega})^{1/2}$ then a typical wave shape for $b^2 > \alpha_h^2$, (i.e. d purely imaginary) is,

$$F(x,y) = W_0\, e^{\pm \sqrt{b^2 - \alpha_h^2}\; y} \begin{cases} \sin \alpha x \\ \cos \alpha x \end{cases} \tag{4.45}$$

If $b^2 < \alpha_h^2$ i.e. d is real then the shape $F(x,y)$ of the internal wave is

$$F(x,y) = W_0 \begin{cases} \sin \sqrt{|b^2 - \alpha_h^2|}\; y \\ \cos \sqrt{|b^2 - \alpha_h^2|}\; y \end{cases} \begin{cases} \sin \alpha x \\ \cos \alpha x \end{cases} \tag{4.46}$$

The magnitude of α_h^2 to be used in these relations is fixed by the mode number m to be $\alpha_{h,m}^2$ which is calculated from a knowledge of a_m^2 by use of Eq. (4.44).
In sum: internal waves are waves of an incompressible fluid which propagate as progressive waves of hydrodynamic velocity whenever the temporal period of such a possible wave exceeds the product of 2π and the reciprocal of the Brunt-Vaisala frequency $N(z) = \left(g\frac{d\rho^0}{dz}\right)^{1/2}$. For a given temporal frequency of such a wave this condition is satisfied at certain depths of the ocean. The (velocity) wave shape in the horizontal plane of the ocean at the appropriate depth can be combinations of: exponential in y, sinusoidal in x; or sinusoidal in y and sinusoidal in x; and can have any non-modal horizontal wave number consistent with the modal wave number in the z direction. The vertical component of velocity wave amplitude at any depth is a modal function of the depth; that is, for a calculated eigenvalue satisfying the boundary conditions one can assign a vertical modal velocity amplitude to the horizontal shape function. The phase velocity of the progressive internal waves of vertical velocity is a function of wavelength, i.e. internal waves exhibit

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frequency dispersion. Since $N(z)$ has one or more maxima in the channel depth the possible progressive internal waves are effectively trapped in certain layers, outside of which the velocity of wave motion decreases exponentially to the boundary values or adjacent layers (see Fig 1). When the temporal frequency of possible internal waves exceeds the Vaisala frequency the internal waves disappear leaving only surface gravity waves propagating at the given temporal frequency. Internal waves are measured as waves of temperature, that is, by use of strings of thermistors.
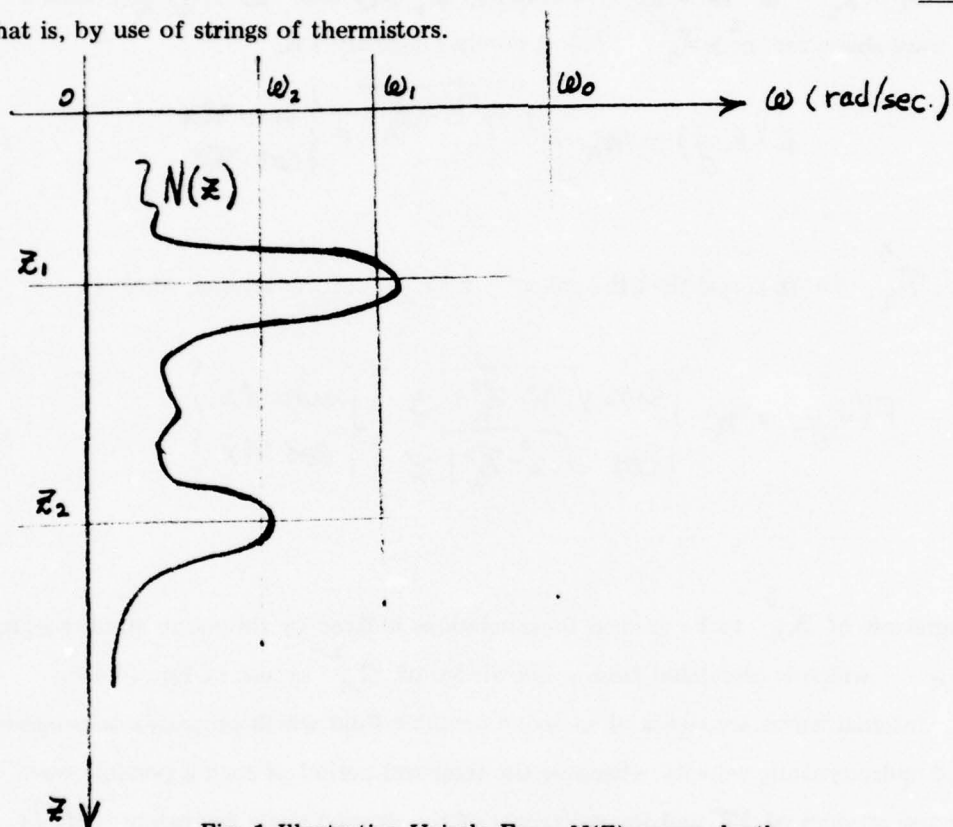
Fig. 1 Illustrative Vaisala Freq. N(z) versus depth.

At (a) ω_0 : no internal waves possible

(b) ω_1 : internal waves present at depth z_1

(c) ω_2 : internal waves present at depth z_2 and z_1 .

General Mathematical Formulation of the Problem of Scalar Waves in a Random Medium

Let $p(\underline{x}, t, q)$ be one realization of a random scalar field in a random medium, and let $N(\underline{x}, t, q)$ be one realization of the refractive coefficient of the random medium itself. The parameter q is a member of a set in which the probability density $P(q)$ is assigned. The statistical moments of p are given by

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$$p^n = \int q^n \, p(\underline{x}, t, q) \, P(q) \, dq \tag{4.47}$$

Thus, to solve a problem of waves in random medium one first solves the problem in one realization of $N(\underline{x}, t, q)$ (thus non-random) of the medium, and then applies a probability distribution to find the statistical moments of the random wave field.

$N(\underline{x}, t, q)$ may be written as $k(\underline{x}, t) \, n(\underline{x}, t, q)$ where k is the propagation constant of the (monochromatic) scalar wave. If $k(\underline{x}, t)$ is constant, the medium is designated as a homogeneous continuous random medium. If $k(\underline{x}, \div)$ is not constant, the medium is an inhomogeneous continuous random medium. If the medium is layered, and each layer has different refractive properties the medium is called a discontinuous random medium.

We consider the case of an inhomogeneous random medium. As will be shown later (see Eq. 5.8, and section 6), the differential equation governing the propagation of acoustic pressure is (to first order),

$$\nabla^2 p - \frac{1}{C^2(\underline{x}, t, q)} \frac{\partial^2 p}{\partial t^2} = \underline{\nabla} \cdot \left[\gamma_\rho (\underline{x}, t, q) \underline{\nabla} p \right] + \gamma_\kappa (\underline{x}, t, q) \frac{1}{C^2(\underline{x}, t, q)} \frac{\partial^2 p}{\partial t^2}$$

$$+ \underline{\nabla} \cdot \left[\eta \underline{\nabla} \cdot \underline{\nabla} \underline{v} + \left(\frac{\eta}{3} + \eta' \right) \underline{\nabla} \, \underline{\nabla} \cdot \underline{v} - 2 \underline{\nabla} \cdot \underline{\underline{R}} \right] \tag{4.48}$$

in which the sound speed C, the compressibility factor γ_κ (of discrete inhomogeneities) and the density factor γ_ρ of discrete inhomogeneities are random functions possessing realization q. In many applications C is approximately deterministic. For these cases the reduced wave equation has the form,

$$\nabla^2 p + k^2(\underline{x}, t) \, p = \underline{\nabla} \cdot \left[\gamma_\rho' (\underline{x}, t, q) \underline{\nabla} p \right] - k^2(\underline{x}, t) \, \gamma_\kappa (\underline{x}, t, q) p \tag{4.49}$$

+ viscous terms + turbulence terms.

Assuming there is a true unit source at $\underline{x}$, we write this equation in the form

$$\nabla^2 p + k^2(\underline{x}, t) \, n(\underline{x}, t, q) p = -\delta(\underline{x}) + F(\underline{x}, t) \tag{4.50}$$

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in which $F(\underline{x}, t)$ is a sum of all homogeneous (i.e. fictitious) sources (such as viscosity effects, etc), and $n(\underline{x}, t, q) = 1 + \gamma_K(\underline{x}, t, q)$.

5. Mathematical Description of Acoustic Inhomogeneities of the Ocean

(Deterministic Case)

The inhomogeneities in the ocean which affect the propagation of sound are predominantly of three types, (1) variable compressibility due to the presence of bubbles, fish, etc., (2) variable density due to thermal gradients, salt concentration, etc., (3) turbulence or random motion of the medium, caused by winds, mixing of currents, thermal gradients, etc. To define these more closely let V_0 be a volume of inhomogeneity, considered initially finite. Outside V_0 the compressibility and density are K, ϱ respectively. Inside V_0 they are K_e, ϱ_e. The turbulent velocity field inside V_0 is $\underline{U}_0$. Following Morse and Ingard [2] we define the nondimensional coefficients γ_K, γ_ϱ of spatial change in compressibility and density as follows,

$$\gamma_K(\underline{x}, t) = \frac{K_e - K}{K} \quad ; \quad \gamma_\varrho(\underline{x}, t) = \frac{\varrho_e - \varrho}{\varrho_e} \tag{5.1}$$

The first order equations of acoustics are given by

$$\frac{\partial \varrho v_i}{\partial t} = -\frac{\partial (c^2 \varrho_1)}{\partial x_i} + \mathcal{V}_i \qquad (N/m^3) \tag{5.2a}$$

$$\frac{\partial \varrho_1}{\partial t} + \frac{\partial (\varrho v_i)}{\partial x_i} = 0 \qquad (NS/m^4) \tag{5.3a}$$

in which ϱ is the background density, ϱ_1, the acoustic (mass) density, v_i' the acoustic particle velocity, C is the sonic speed and $\mathcal{V}_i$ are viscous and turbulence terms (to be supplied later). We first assume that background density and sonic speed, are slowly varying functions of $\underline{x}$, t and that the acoustic pressure $p \approx c^2 \varrho_1$. Thus,

$$\frac{\partial v_i}{\partial t} = -\frac{1}{\varrho} \frac{\partial p}{\partial x_i} + \frac{1}{\varrho} \mathcal{V}_i \tag{5.2b}$$

$$\frac{\partial v_i}{\partial x_i} = -\frac{1}{\varrho c^2} \frac{\partial p}{\partial t} \tag{5.3b}$$

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Taking the time derivative of 5.3b and the divergence of 5.2b, then eliminating $\partial^2 v_i / \partial t \partial x_i$ one arrives at the wave equation,

$$-\frac{\partial}{\partial x_i}\left[\frac{1}{\varrho}\frac{\partial p}{\partial x_i}\right] + \frac{\partial}{\partial t}\left[\frac{1}{\varrho c^2}\frac{\partial p}{\partial t}\right] + \frac{\partial}{\partial x_i}\left(\frac{1}{\varrho}v_i\right) = 0 \qquad (5.4)$$

Writing ϱ_e, c_e for location points inside inhomogeneities, and ϱ_o, c_o for points outside, one obtains by subtraction,

$$-\frac{\partial}{\partial x_i}\left[\left(\frac{1}{\varrho_e} - \frac{1}{\varrho_o}\right)\frac{\partial p}{\partial x_i}\right] + \frac{\partial}{\partial t}\left[\left(\frac{1}{\varrho_e c_e^2} - \frac{1}{\varrho_o c_o^2}\right)\frac{\partial p}{\partial t}\right] + \frac{\partial}{\partial x_i}\left[\frac{v_{ie}}{\varrho_e} - \frac{v_{io}}{\varrho_o}\right]$$

$$= \frac{\partial}{\partial x_i}\left(\frac{1}{\varrho_o}\frac{\partial p}{\partial x_i}\right) - \frac{\partial}{\partial t}\left(\frac{1}{\varrho_o c_o^2}\frac{\partial p}{\partial t}\right) - \frac{\partial}{\partial x_i}\left(\frac{v_{io}}{\varrho_o}\right)$$

$$(5.5)$$

Neglecting the <u>change</u> in viscosity and assuming all time scales to be fast time, we obtain,

$$\frac{\partial^2 p}{\partial x_i \partial x_i} - \frac{1}{c_o^2}\frac{\partial^2 p}{\partial t^2} = \frac{\partial}{\partial x_i}\left[\gamma_\varrho(\underline{x},t)\frac{\partial p}{\partial x_i}\right] + \frac{1}{c_o^2}\frac{\partial}{\partial t}\left(\gamma_K(\underline{x},t)\frac{\partial p}{\partial t}\right)$$

$$+ \frac{\partial v_i}{\partial x_i} \qquad (5.6)$$

in which we have used the definitions $K^{-1} = \varrho c^2$, $\gamma_K = (K_e - K_o)/K$, $\gamma_\varrho = (\varrho_e - \varrho)/\varrho_e$

. The quantity v_i remains to be identified. It is conventional to include in it first order viscosity terms, and the Reynolds stress tensor R_{ij}, i.e.

$$v_i \equiv \eta \frac{\partial}{\partial x_i}\left(\frac{\partial v_i}{\partial x_j}\right) + \left(\frac{\eta}{3} + \eta'\right)\frac{\partial}{\partial x_i}\frac{\partial v_j}{\partial x_j} - 2\frac{\partial}{\partial x_j}R_{ij} \qquad (5.7)$$

in which η, η' are the shear and dilatational viscosities respectively. The term R_{ij} is discussed below. Substitution of the explicit form of v_i into 5.7 leads to the result that

$$\frac{\partial^2 p}{\partial x_i \partial x_i} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = \frac{\partial}{\partial x_i}\left[\gamma_s(\underline{x},t)\frac{\partial p}{\partial x_i}\right] + \gamma_\kappa(\underline{x},t)\frac{\partial^2 p}{c^2 \partial t^2}$$

$$-\frac{\partial}{\partial x_i}\left\{\eta\frac{\partial}{\partial x_j}\frac{\partial v_i}{\partial x_j} + \left(\frac{\eta}{3}+\eta'\right)\frac{\partial}{\partial x_i}\frac{\partial v_j}{\partial x_j} - 2\frac{\partial}{\partial x_j}R_{ij}\right\} \quad (5.8)$$

R is a matrix with elements

$$R_{ij} = \frac{1}{2}\varrho\, u_i u_j, \qquad\qquad u_i = U_i + v_i \quad (5.9)$$

Equation 5.8 is the form given by Morse and Ingard [3]. It expresses the scattering of the acoustic pressure (l.h.s.) due to volume inhomogeneities in the medium (r.h.s.). These inhomogeneities are of the monopole (1st r.h.s.), dipole (2nd r.h.s.) and quadrupole (3rd r.h.s.) type. In the presence of acoustic processes the velocity u_i is the sum of a flow velocity $\underline{U}$ and an acoustic particle velocity $\underline{V}$. Hence

$$R_{ij} = \frac{1}{2}\varrho\left(U_i U_j + v_i U_j + v_j U_i + v_i v_j\right) \quad (5.10)$$

The term $U_i U_j$ represents the generation of sound by turbulence. In a pure scattering problem it is omitted. The term $v_i v_j$ is a nonlinear term (viz., sound interacting with sound). The scattering of the sound by gross flow is contained in the terms $v_i U_j$.

The solution of Eq. 5.8 is best carried out by use of an appropriate Green's function g. Choosing unbounded space, we seek solutions only in the far field, i.e., we choose

$$\lim_{|\underline{x}|\to\infty} g(\underline{x},t\,|\,\underline{x}_0,t_0) = \frac{1}{4\pi|\underline{x}|}\delta\left(t_0 - t + \frac{|\underline{x}|}{c} - \frac{\underline{x}\cdot\underline{a}_\kappa}{c}\right) \quad (5.11)$$

The coordinates of the volume V_0 which is causing the scattering are thus $\underline{x}_0$, t_0. The scattered (far) field of pressure ($p_{s\kappa}$) due to an inhomogeneity of compressibility is thus,

$$p_{s\kappa}(\underline{x},t) = \int_{V_0} \gamma_\kappa(\underline{x}_0,t_0)\frac{\partial^2 p(x_0,t_0)}{c^2 \partial t^2}\frac{1}{4\pi|\underline{x}|}\delta\left(t_0 - t + \frac{|\underline{x}|}{c} - \frac{\underline{x}\cdot\underline{x}_0}{c}\right)dV_0$$

$$(5.12)$$

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The total pressure p at far-field point $\underline{X} = (x, y, z)$ due to an incident pressure p_{INC} is,

$$p(\underline{x},t) = p_{INC}(\underline{x},t) + \iiiint d^3x_0 dt_0 \frac{1}{c^2} \gamma_K(\underline{x}_0,t_0) \frac{\partial^2 p}{\partial t_0^2}(\underline{x}_0,t_0) g(\underline{x},t|\underline{x}_0,t_0)$$

(5.13)

Equation 5.13 is a Fredholm integral equation of the second type. To solve it we require a knowledge of the second time derivative of the acoustic pressure throughout the scattering volume. In a similar way the scattering due to density variations and turbulence may be expressed in the form

$$\left\{ \mathcal{I} \delta(\underline{x}-\underline{x}_0, t-t_0) - \lambda \, \mathcal{R}(\underline{x}, t|\underline{x}_0,t_0) \right\} p(\underline{x}_0,t_0) = p(\underline{x},t) \quad (5.14)$$

in which $\mathcal{I}$ is the idemfactor, $\mathcal{R}$ is an integral operator and λ is a parameter. The three forms of $\mathcal{R}$ corresponding to the three types of inhomogeneity are

$$\left.\begin{matrix} \mathcal{R}_K \\ \mathcal{R}_\varrho \\ \mathcal{R}_\Sigma \end{matrix}\right\} = \iiiint dt_0 \, d^3\underline{x}_0 \, \frac{\delta\left(t_0 - t + \frac{|\underline{x}|}{c} - \frac{\underline{x}_0 \cdot \underline{q}_k}{c}\right)}{4\pi |\underline{x}|} \left\{\begin{matrix} \gamma_K(\underline{x}_0,t_0) \frac{1}{c^2}\frac{\partial^2}{\partial t^2} & (5.15) \\[2mm] \underline{\nabla} \cdot \left[\gamma_\varrho(\underline{x}_0,t_0) \nabla \right] & (5.16) \\[2mm] (-) 2 \underline{\nabla} \cdot \varrho \, \underline{v}(\underline{x}_0,t_0) \underline{U}(\underline{x}_0,t_0) \cdot \nabla & (5.17) \end{matrix}\right.$$

In Eq. 5.17 the acoustic particle velocity $\underline{v}$ is a function of acoustic pressure i.e., $\underline{v} = f(p)$. The solution of Eq. 5.14 in the general case is seen to be an extremely difficult task to undertake. Current practice is to simplify all operations by use of the single scattering approximation (viz., $p(\underline{x}_0,t_0) = p_{INC}(\underline{x}_0,t_0)$), together with the choice of plane wave incidence (viz., $p_{INC} = A \exp i (k_i x - k_i ct)$)). This simplification leads to the conclusion that the scattered field is proportional to the four-

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dimensional Fourier transform of the analytic form of the inhomogeneity. A fuller discussion of these simplifications is found in Sect. 18.

6. Monin's Equation and Allied Equations (Rytov, Parabolic)

The dipole term in Eq. 5.8 represents an inhomogeneity of the medium due to a change in density. In the ocean such changes in density are caused by changes in temperature, salinity and depth. These changes in density change the speed of sound. A convenient mathematical description of this relation consists in expanding the sound speed in powers of these changes.

Often the first (or linear) term in such expansions suffice. We consider here only the linear term and choose temperature alone as a single representative variable. If the local fluid temperature is T it is known [4] that the local sound speed C is given by,

$$C = N\sqrt{T}, \qquad\qquad N = \text{const.} \qquad\qquad (6.1)$$

Thus,

$$dC = \frac{N\,dT}{2\sqrt{T}} \qquad\qquad (6.2)$$

Multiplying by $\sqrt{T_0}/\sqrt{T_0}$, and reducing the result, leads to the statement that

$$\Delta C \approx C_0 \Delta T / 2 T_0 \qquad\qquad (6.3)$$

in which $\sqrt{TT_0} \approx T_0$. The local sound speed in a medium with fluctuating temperature $T'(=\Delta T)$ is described to first order in small temperature changes by the formula

$$C = C_0 + \frac{C_0}{2T} T' \qquad\qquad (6.4)$$

Upon squaring and then rejecting the quantity in T'^2, one obtains the convenient result that

$$C^2 = C_0^2 \left(1 + \frac{T'}{T_0}\right), \qquad C_0^2 = N^2 T_0 \qquad\qquad (6.5)$$

We use this equation in the following analysis. Now to first order in acoustic quantities, the acoustic pressure p_a is given by

$$p_a = c_o^2 \rho_a = \rho_o c_o^2 \left(\rho_a / \rho_o \right) \tag{6.6}$$

where ρ_a is the first order increment of density. It is convenient to define a dimensionless pressure Π by the relation

$$\Pi = \frac{p_a}{\rho_o c_o^2} = \frac{\rho_a}{\rho_o} \tag{6.7}$$

Neglecting viscosity, thermal gradients and convection the sonic particle velocity associated with the acoustic pressure p_a is given by

$$\rho \frac{\partial v}{\partial t} = - \nabla p_a \tag{6.8}$$

Setting $\rho_o \approx \rho$, we obtain from this the (steady state) relation

$$v_i \approx \frac{c_o^2}{i \omega} \frac{\partial \Pi}{\partial x_i} , \qquad \frac{d}{dt} = - i \omega \tag{6.9}$$

We next turn to Eq. 5.8 and consider p to be the acoustic pressure due to density changes and turbulence. Substituting Eq. 6.9 and 6.7 into Eq. 5.8 and neglecting compressibility effects we find (for the steady state) that

$$\frac{\partial^2 \Pi}{\partial x_j \partial x_j} + k^2(\underline{x},t) \Pi = \frac{\partial}{\partial x_i} \left[\gamma_k (\underline{x},t) \frac{\partial \Pi}{\partial x_i} \right] - \frac{d}{i \omega} \frac{\partial^2}{\partial x_i \partial x_j} \left(U_i \frac{\partial \Pi}{\partial x_j} \right) \tag{6.10}$$

The change in density due to temperature fluctuations can be related to the parameter γ_ρ . We write

$$\gamma_\rho = - \frac{(\rho - \rho_e)}{\rho_e} = - \frac{\Delta \rho}{\rho_e} \tag{6.11}$$

To first order in the temperature fluctuations the following relation is assumed to hold,

$$\frac{\Delta \rho}{\rho_e} = \frac{T'}{T_o} \tag{6.12}$$

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Thus, Eq. (12) reduces to

$$\frac{\partial^2 \Pi}{\partial x_j^2} + k^2(\underline{x},t)\Pi = -\frac{\partial}{\partial x_i}\left[\frac{T'}{T_0}\frac{\partial \Pi}{\partial x_i}\right] - \frac{2}{i\omega}\frac{\partial^2}{\partial x_i \partial x_j}\cdot\left(U_i\frac{\partial \Pi}{\partial x_j}\right) \qquad (6.13)$$

When changes in depth d and in salinity S affect the density of the medium we can write,

$$\frac{\Delta\varrho}{\varrho} \sim \alpha \frac{d'}{d_0} \qquad ; \qquad \frac{\Delta\varrho}{\varrho} \sim \beta \frac{S'}{S_0} \qquad (6.14)$$

in which d_0, S_0 are reference quantities and α, β are constants. The propagation of sound in the ocean in the presence of fluctuations of velocity, and fluctuations of density due to temperature gradients, salinity and depth of signal plus fluctuations in compressibility is therefore governed by the steady state equation in first order quantities,

$$\frac{\partial^2 \Pi}{\partial x_j^2} + k^2(\underline{x},t)\Pi = -k^2(\underline{x},t)\gamma_\kappa(\underline{x},t)\Pi - \frac{\partial}{\partial x_i}\left[\left(\frac{T'}{T_0} + \alpha\frac{d'}{d_0} + \beta\frac{S'}{S_0}\right)\frac{\partial \Pi}{\partial x_i}\right]$$

$$- \frac{2}{i\omega}\frac{\partial^2}{\partial x_i \partial x_j}\cdot\left(U_i\frac{\partial \Pi}{\partial x_j}\right) \qquad (6.15)$$

Eq. 6.13 is the equation of Monin [6]. Eq. 6.15 can be used to show the effect of compressibility alone. In the absence of external sources this case can be described by the equation,

$$\frac{\partial^2 \Pi}{\partial x_j^2} + \frac{\omega^2}{\mathscr{C}^2(\underline{x},t)}\Pi = 0 \qquad (6.16)$$

in which

$$\mathscr{C}^2(\underline{x},t) = \frac{c^2(\underline{x},t)}{1 + \gamma_k(\underline{x},t)} \qquad (6.17)$$

An alternative derivation of the appropriate equation governing the propagation of sound in the ocean is that of Neubert and Lumley [8]. They begin with Eqs. 5.3 and 5.4 in which they make the substitutions

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$$\varsigma = \varsigma + \varsigma' \tag{6.18}$$

$$\underline{u} = \underline{U} + \underline{v}' \tag{6.19}$$

$$p = P_0 + p' \tag{6.20}$$

in which the primed symbols are acoustic quantities. Thus the equation of continuity leads to the statement that

$$\frac{\partial \varsigma'}{\partial t} + \frac{\partial (\varsigma_0 U_i)}{\partial x_i} + \frac{\partial (\varsigma' U_i)}{\partial x_i} + \frac{\partial (\varsigma v_i')}{\partial x_i} + \frac{\partial (\varsigma_0 v_i')}{\partial x_i} = 0 \tag{6.21}$$

Assuming the scaling laws 8.13 through 8.17 hold (see below), and that

$$\frac{\varsigma'}{\varsigma_0} \ll 1 \tag{6.22}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \qquad \text{(i.e., incompressible flow)} \tag{6.23}$$

they then neglect as negligible the terms $\partial (\varsigma' v_i')/\partial x_i$ (relative to $\partial (\varsigma_0 v_i')/\partial x_i$ and $\varsigma' \partial U_i/\partial x_i$. The result is,

$$\frac{\partial \varsigma'}{\partial t} + U_i \frac{\partial \varsigma'}{\partial x_i} + \frac{\partial (\varsigma_0 v_i')}{\partial x_i} = 0 \tag{6.24}$$

This equation is properly scaled if

$$\varsigma' c = \mathcal{O}(\varsigma_0 |\underline{v}'|) \tag{6.25}$$

$$\varsigma' |\underline{U}| = \mathcal{O}(\varsigma_0 |\underline{v}'|) \tag{6.26}$$

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The Navier-Stokes equation (= Eq. 5.2) is similarly expanded in equilibrium quantities and fluctuating (or acoustic) quantities. The result is,

$$\frac{\partial}{\partial t}\left[\varrho_0 U_i + \varrho' U_i + \varrho_0 v_i' + \varrho' v_i'\right] + \frac{\partial}{\partial x_j}\left[\varrho_0 U_i U_j + \varrho' U_i U_j + \varrho_0 v_i' U_j\right.$$

$$\left. + \varrho' v_i' U_j + \varrho_0 U_i v_j' + \varrho' U_i v_j' + \varrho_0 v_i v_j + \varrho' v_i' v_j'\right]$$

$$= -\frac{\partial P'}{\partial x_i} + \eta \frac{\partial}{\partial x_i}\left[\frac{\partial U_i}{\partial x} + \frac{\partial v_i'}{\partial x_i}\right] + \left(\frac{\eta}{3} + \eta'\right)\frac{\partial}{\partial x_i}\left[\frac{\partial U_j}{\partial x_j} + \frac{\partial v_j'}{\partial x_j}\right]$$

$$(6.27)$$

Assuming that the gross flow field $\underline{U}$ is incompressible and irrotational and that the viscous damping of the sound wave is negligible one drops all terms in the viscosity coefficients. In terms of the scaling equations 8.13 through 8.17 (see below) the first two terms on the l.h.s. scale as

$$\frac{\partial \varrho_0 U_i}{\partial t} + \varrho' \frac{\partial U_i}{\partial t} + \frac{\partial \varrho'}{\partial t} U_i \rightarrow \frac{\varrho_0 U^2}{L_u} + \frac{\varrho' c U}{\lambda}$$

$$(6.28)$$

(Note the second term l.h.s. drops out relative to the first term l.h.s.). Now if we choose U such that

$$\frac{U^2}{L_u} \ll \frac{c^2}{\lambda^2}$$

$$(6.29)$$

then

$$\frac{\varrho_0 U^2}{L_u} \ll \frac{\varrho' c U}{\lambda}$$

$$(6.30)$$

Upon this choice we drop $\varrho_o \, \partial U_i / \partial t$ relative to $(\partial \varrho' / \partial t) U_i$. If further we write

$$\varrho_o \, V_i' = \mathcal{O}(\varrho'' U_i)$$

then the 3rd term l.h.s. has the same scale as $(\partial \varrho' / \partial t) U_i$ and is thus retained. The 4th term $\partial/\partial t \, (\varrho' V_i')$ is of second order relative to $\partial/\partial t (\varrho_o V_i')$ and is dropped. The fourth term r.h.s. requires careful analysis. Neglecting non-acoustic coupled terms (i.e., $\varrho_o \, U_i \, U_j'$), and terms of second order in acoustic quantities (i.e., $\varrho' V_i' U_j'$, $\varrho_o V_i' V_i'$), and observing the condition that the fluid is incompressible (= Eq. 6.23), one reduces the divergence of $\varrho u_i u_j'$ to the set

$$\varrho'_{,j} \, U_i \, U_j + \varrho' \, U_{i,j} \, U_j + \varrho_o \, V_{i,j}' \, U_j + \varrho_o \, U_{i,j} \, V_j' + \varrho_o \, U_i \, V_{j,j}'$$

By grouping terms, then using Eq. 6.24 to eliminate quantities, one reduces Eq. 6.27 to the result

$$\frac{D}{Dt} (\varrho_o \, V_i') + \frac{\partial P'}{\partial x_i} + \frac{\partial U_i}{\partial x_j} \varrho_o \, V_j' = -\varrho' \, U_i \, \frac{\partial U_i}{\partial x_i} \qquad (6.31)$$

Taking the time derivative of Eq. 6.24 and the gradient of Eq. 6.31, then eliminating the term $\dfrac{D^2}{Dt \, \partial x_i} (\varrho_o \, V_i')$ yields the result

$$\frac{\partial^2 P'}{\partial x_i^2} - \frac{D^2 \varrho'}{Dt^2} = -2 \, \frac{\partial U_i}{\partial x_j} \, \frac{\partial}{\partial x_i} (\varrho_o \, V_j') - \frac{\partial}{\partial x_i} (\varrho' \, U_j) \, \frac{\partial U_i}{\partial x_j} \qquad (6.32)$$

Now in the linear equation of state $P' = \varrho' c^2$ we assume both ϱ' and c are functions of $\underline{x}$, t. Thus

$$\frac{D \varrho'}{Dt} = \frac{1}{c^2} \frac{DP'}{Dt} + \varrho' c^2 \left\{ \left(\frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x_i} \right) \frac{1}{c_2(\underline{x},t)} \right\} \qquad (6.33)$$

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For velocity turbulence the characteristic length L_u (see Eq. 8.11) is much larger than Λ_ε . Since the parameter c depends on the turbulence of the medium, we scale as follows:

$$\frac{\partial}{\partial t} \frac{1}{c^2} \ll U \frac{\partial}{\partial x_i} \frac{1}{c^2} \qquad \text{corresponds to} \qquad \frac{U}{L_u c^2} \ll \frac{U}{\Lambda_\varepsilon c^2} \tag{6.34}$$

Using this scaling and Eq. 6.24 it is seen that

$$\frac{D\varsigma'}{\partial t} = -2 \frac{\partial (\varsigma_c V_d)}{\partial x_c} \frac{U_i}{c} \frac{\partial c}{\partial x_i} - \frac{1}{c^2} \left(\frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x_i} \right) P' \tag{6.35}$$

Taking the time derivative once more, one obtains

$$\frac{D^2 s'}{Dt^2} = \frac{1}{c^2} \frac{D^2 P'}{Dt^2} + \frac{DP'}{Dt} \frac{D}{Dt}\left(\frac{1}{c^2}\right) + \frac{D\varsigma'}{Dt}\left[c^2 U_i \frac{\partial}{\partial x_i} \left(\frac{1}{c^2}\right) \right]$$

$$+ \varsigma' \frac{D}{Dt}\left[c^2 U_i \frac{\partial}{\partial x_i} \left(\frac{1}{c^2}\right) \right] \tag{6.36}$$

The third term r.h.s. scales as $c \varsigma' (U/\lambda \Lambda_\varepsilon)$ while the fourth term scales as $(c \varsigma' U / \lambda \Lambda_\varepsilon)$ $\frac{U\lambda}{L_u C}$. Since $U\lambda/L_u C \ll 1$ the fourth term is negligible relative to the third. We next compare the first and second term r.h.s. The first term scales as P'/λ^2 while the second term scales as $(P'/\lambda^2)(U\lambda/c L_u)$. Since $(U\lambda/c L_u) \ll 1$ we neglect the second term. Using all results it is seen that

$$\frac{D^2 \varsigma'}{Dt^2} = \frac{1}{C^2}\left(\frac{D}{Dt}\right)^2 (P') + 2\varsigma_0 \frac{\partial v_j'}{\partial x_j} \frac{U_i}{C} \frac{\partial c}{\partial x_i} \tag{6.37}$$

Substituting this into Eq. 6.27 and observing all conditions it is seen that stochastic wave equation in a medium with convection velocity U_i' , reduces to

$$\frac{\partial^2 P}{\partial x_i^2} - \frac{1}{C^2}\left(\frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x_i}\right)^2 P' = 2\left(\frac{\partial \varsigma_0 v_j'}{\partial x_j}\right) \frac{U_i}{C} \frac{\partial c}{\partial x_i}$$

$$- \frac{\partial U_i}{\partial x_j} \frac{\partial}{\partial x_i}\left(2\varsigma_0 v_j' - \varsigma' U_j\right) \tag{6.38}$$

$$C = C(\underline{x}, t)$$

This is the equation of Neubert and Lumley [7]. The l.h.s. represents the conventional linear small amplitude propagation of sound in a medium with sonic speed $C(\underline{x}, t)$. The first term on the r.h.s. couples soundspeed gradients (due to temperature and density variations) with turbulent velocity and the divergence of acoustic particle velocity. The second term couples the shear turbulent velocities with acoustic quantities. The orders of magnitude of the terms in Eq. 6.38 are as follows

$$\frac{\partial^2 P'}{\partial x^2} = \frac{\partial^2 \varsigma_0 C v'}{\partial x^2} = \mathcal{O}\left(\frac{\varsigma_0 C v'}{\lambda^2}\right) \tag{6.39}$$

$$\frac{\partial \varsigma_0 v'}{\partial x}\left(\frac{U}{C}\right)\frac{\partial c}{\partial x} = \mathcal{O}\left(\frac{\varsigma_0 v' U}{\lambda \lambda_g}\right) \tag{6.40}$$

$$\frac{\partial U}{\partial x} \frac{\partial}{\partial x}\left(2\varsigma_c v'\right) = \mathcal{O}\left(\frac{\varsigma_0 v' U}{\lambda \lambda_g}\right) \tag{6.41}$$

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If the r.h.s. of Eq. 6.38 is to be negligible relative to the l.h.s. then

$$\frac{\varrho_o v' U}{\lambda \Lambda_c} \ll \frac{\varrho_o c v'}{\lambda^2} , \qquad \text{or} \qquad \left(\frac{U}{c}\right) \frac{\lambda}{\Lambda_c} \ll 1 \qquad (6.42)$$

This is certainly satisfied if

$$\frac{U}{c} \ll 1 \qquad \text{as well as} \qquad \frac{\lambda}{\Lambda_c} \ll 1 \qquad (6.43)$$

The effect of the convection term $U \frac{\partial}{\partial x}$ can be neglected relative to $\partial / \partial t$ if

$$\frac{U}{\lambda} \ll \frac{c}{\lambda} \qquad \text{or} \qquad \frac{U}{c} \ll 1 \qquad (6.44)$$

If the scattering by inhomogeneities is due to temperature gradients rather than velocity turbulence then Λ_g in Eq. 3.40 is replaced by Λ_θ , and Eq. (6.42) reads,

$$\left(\sigma_K\right)^{\frac{1}{2}} \frac{U}{c} \frac{\lambda}{\Lambda_g} \ll 1 , \qquad \sigma_K = \frac{\Lambda_c^2}{\Lambda_\theta^2} \qquad (6.45)$$

provided $\lambda \ll \vartheta(\Lambda_g)$. Supposing

$$\lambda > \vartheta(\Lambda_g) \qquad (6.46)$$

then the condition given by Eq. (6.45) can be modified to read

$$\sigma_K^{\frac{1}{2}} \left(\frac{U}{c}\right) \frac{\lambda^2}{\Lambda_g^2} \ll \frac{\lambda}{\Lambda_g} \qquad (6.47a)$$

This is certainly true if

$$\left(\sigma_K\right)^{\frac{1}{2}} \left(\frac{U}{c}\right) \frac{\lambda^2}{\Lambda_g^2} \ll 1 \qquad (6.47b)$$

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Thus if turbulent inhomogeneities are present in the ocean one can use the reduced equation of wave propagation,

$$\frac{\partial^2 P'}{\partial x_i^2} - \frac{1}{c^2}\frac{\partial^2 P'}{\partial t^2} = 0 \quad , \qquad C = C(\underline{x}, t, \omega) \; ; \; P' = P'(\underline{x}, t, \omega)$$

(6.48)

provided the various conditions given by Eqs. (6.44), (6.45), (6.46) and (6.47) are satisfied in appropriate groupings.

It is noted that both C and P are considered functions of position, time, and spectral frequency. If the chromatic assumption is made, ω and t appear in Eq. (6.48) as the product ωt, and ω and c appear as a ratio $k = \omega/c = k_0 \mu(\underline{x})$ where μ is the (random) index of refraction. Spectral broadening due to interaction between the acoustic wave and the scattering inhomogeneity is neglected and Eq. (6.48) reduces to the stochastic Helmholtz equation,

$$\nabla^2 p(\underline{x}) + k_0^2 \mu(\underline{x})^2 p(\underline{x}) = 0$$

(6.49)

Equations of Propagation Reported by Chernov

The equation of continuity for a medium of density ϱ which has a spatially (though not temporally) varying background (or equilibrium) mass density ϱ_0 is known to first order in the acoustic Mach number (= v/c) to be

$$\frac{\partial \varrho_1}{\partial t} + \nabla \varrho_0 \cdot \underline{v} = - \varrho_0 \underline{\nabla} \cdot \underline{v}_1 \qquad\qquad \varrho = \varrho_0 + \varrho_1 \quad (6.50)$$

From the linear theory of the acoustic potential it is known that the acoustic pressure to first order in v/c, obeys the equations,

$$- \varrho_0 c^2 \underline{\nabla} \cdot \underline{v}_1 = \frac{\partial p_1}{\partial t}$$

(6.51)

$$\frac{\partial^2 \varrho_1}{\partial t^2} = \nabla^2 p_1$$

(6.52)

$$\frac{\partial \underline{v}_1}{\partial t} = - \frac{\nabla p_1}{\varrho_0}$$

(6.53)

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Substituting (6.51) into (6.50), then differentiating with respect to t, then using (6.52)
and (6.53), one arrives at

$$\frac{1}{c^2} \frac{\partial^2 p_1}{\partial t^2} - \nabla^2 p_1 + \left(\underline{\nabla} \log \varrho_0 \right) \cdot \underline{\nabla} p_1 = 0 \qquad (6.54)$$

Now let $\Delta \varrho$ and Δc be small random deviations from ϱ_0 and c_0 respectively. Then
Eq. (6.54) can be replaced by the equation

$$\frac{1}{c_0^2} \frac{\partial^2 p_1}{\partial t^2} - \nabla^2 p_1 = \frac{2 \Delta c}{c_0^3} \frac{\partial^2 p_1}{\partial t^2} - \frac{1}{\varrho_0} \underline{\nabla} \left(\Delta \varrho \right) \cdot \underline{\nabla} p_1 \qquad (6.55)$$

This is the (Rayleigh) wave equation reported by Chernov [9] by the method of perturbations. Written in this way the r.h.s. becomes a statement that the local acoustic pressure
p_1 interacts with inhomogeneities created by changes in sonic speed (= Δc) and equilibrium density (= $\Delta \varrho$) and thus becomes a source of scattered waves (= p_1 on the l.h.s).
This equation is analogous to Eq. (5.8) in that the first term on the r.h.s. represents a
source due to inhomogeneities in compressibility, and the second term a source due to
inhomogeneities in mass density. Differences however do appear. The density term in
Eq. (5.8) has the form

$$\underline{\nabla} \gamma_\varrho \cdot \underline{\nabla} p + \gamma_\varrho \nabla^2 p \qquad (6.56)$$

By contrast Eq. (6.55) neglects the term $\gamma_\varrho' \nabla^2 p$ as negligible compared to the first term
in (6.56). In addition the compressibility factor is

$$\frac{K_e - K}{K} \qquad \text{for Eq. 2.8,} \qquad \frac{2 \Delta c}{c_0} \qquad \text{for Eq. (3.55)}$$

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The physical cause of ΔC and $\Delta \varrho$ is taken by Chernov to be fluctuations in temperature. Thus he writes

$$\Delta C = \left(\frac{\partial C}{\partial T}\right)_{p_0} \Delta T \quad ; \quad \Delta \varrho = \left(\frac{\partial \varrho}{\partial T}\right)_{p_0} \Delta \varsigma \qquad (6.57)$$

in which p_0 indicates constant pressure. For sonic processes at deep submergence (= high hydrostatic pressure), these formulas must be modified to read,

$$\Delta C = \left(\frac{\partial C}{\partial T}\right)_{p_0} \Delta T + \left(\frac{\partial C}{\partial p_0}\right)_T \Delta p_0 \qquad (6.58)$$

$$\Delta p = \left(\frac{\partial p}{\partial T}\right)_{p_0} \Delta T + \left(\frac{\partial C}{\partial p_0}\right)_T \Delta p_0 \qquad (6.59)$$

A comprehensive form of Eq. (6.55) may be constructed from (6.58) and (6.59). It is

$$\frac{1}{C_0^2} \frac{\partial^2 p_1}{\partial t^2} - \nabla^2 p_1 = \frac{2}{C_0^3} \left\{ \left(\frac{\partial C}{\partial T}\right)_{p_0} \Delta T + \left(\frac{\partial C}{\partial p_0}\right)_T \Delta p_0 \right\} \frac{\partial^2 p_1}{\partial t^2}$$

$$- \frac{1}{\varsigma} \nabla \left\{ \left(\frac{\partial \varrho}{\partial T}\right)_{p_0} \Delta T + \left(\frac{\partial C}{\partial p_0}\right)_T \Delta p_0 \right\} \cdot \nabla p_1 \qquad (6.60)$$

This equation now describes the scattering of acoustic waves by fluctuations in background of equilibrium temperature and pressure.

Rytov's Equation

If inhomogeneities in density are neglected then one may write Eq. (6.60) in the form

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$$\frac{(1+\mu)^2}{C_0^2} \frac{\partial^2 p_1}{\partial t^2} - \nabla^2 p_1 = 0 \qquad (6.61)$$

in which $\mu(x)$ is the random deviation of the refractive index from its mean value of unity with $|\mu| \ll 1$. In Rytov's method one seeks a solution of the form

$$p = A_0 \exp\left[-i\omega t + i\,\psi(x)\right]$$

$$\psi = \mathcal{R}e\,\psi + i\,\mathcal{I}m\,\psi \qquad (6.62)$$

Direct substitution of (6.62) into (6.61) leads to an equation in ψ, viz,

$$(\nabla\psi)^2 - i\,\nabla^2\psi = k^2 n^2, \qquad k = \frac{\omega}{c}, \quad n = \frac{C(1+\mu)}{C_0} \qquad (6.63)$$

It is noted that the source term is inhomogeneous. This nonlinear equation can be linearized by expanding ψ in a series of small perturbations,

$$\psi = \psi_0 + \psi' + \cdots, \qquad (6.64)$$

in which ψ_0 ($= \underline{k} \cdot \underline{x}$) satisfies the wave equation for the homogeneous medium, ψ' is a small perturbation on ψ_0, etc. Substituting (6.64) into (6.63) and noting that

$$\frac{1}{k}\nabla\psi' = \vartheta(\mu), \qquad \frac{1}{k}|\nabla\psi'| \ll 1 \qquad (6.65)$$

one obtains the linear equation

$$2(\nabla\psi_0 \cdot \nabla\psi') - i\,\nabla^2\psi' = 2\mu k^2 \qquad (6.66)$$

Assuming the incident (= unperturbed) wave travels in the x-direction so that $\psi_0 = kx$, one obtains the linearized Rytov equation

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$$2 \left(\underline{\nabla} \psi_0 \cdot \underline{\nabla} \psi' \right) - i \nabla^2 \psi' = 2 \mu k^2 \tag{6.67}$$

Other forms of the (nonlinear) Rytov equation are in use. Writing

$$\underline{\nabla} \psi = \frac{\partial \psi}{\partial x} \hat{i} + \underline{\nabla}_\perp \psi , \qquad \nabla_\perp = \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \tag{6.68}$$

and assuming that

$$\left| \frac{\partial^2 \psi}{\partial x^2} \right| << \nabla_\perp^2 \psi \tag{6.69}$$

one arrives at

$$2k \frac{\partial \psi'}{\partial x} - i \nabla_\perp^2 \psi' + \left(\nabla_\perp \psi' \right)^2 = 2 \mu k^2 \tag{6.70}$$

This is the basic equation of the Rytov method. It has complex coefficients, is non-linear and has an inhomogeneous source term. In all forms (as given by Eq. (6.63), (6.67) and (6.70) the Rytov equation is related to the ray equations of geometric optics. To find this relation we return to Eq. (6.61) and reduce it to the steady state Helmholtz equation at frequency ω_0 , i.e.

$$\left[\nabla^2 + k_0^2 n^2 (\underline{x}) \right] p_i (\underline{x}) = 0 , \qquad k_0 = \frac{\omega_0}{c_0} , \quad n^2 = (1+\mu)^2 \tag{6.71}$$

At high frequency $\omega_0 = k_0 c_0 = 2\pi c_0 / \lambda_0$, one can construct an asymptotic solution of the form

$$p(\underline{x}) \sim e^{i k_0 \psi(\underline{x})} \sum_{m=0}^{\infty} \frac{p_m (\underline{x})}{(i k_0)^m} \tag{6.72}$$

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Upon substitution of (6.72) into (6.71) one finds that

$$\left\{ (ik_0)^2 \left[(\underline{\nabla}\psi)^2 - n^2 \right] + ik_0 \left[\underline{\nabla}\cdot\nabla\psi + 2\underline{\nabla}\psi\cdot\underline{\nabla} \right] + \nabla^2 \right\}$$

$$\times \sum_{m=0}^{\infty} \frac{p_m(\underline{x})}{(ik_0)^n} = 0 \tag{6.73}$$

Since each coefficient in this expansion must vanish it is seen that the following conditions hold under which (6.73) is valid,

Term	Condition
k_0^2	$(\underline{\nabla}\psi)^2 = n^2$ $\qquad$ (6.74)
k_0	$\left(\underline{\nabla}\cdot\nabla\psi + 2\underline{\nabla}\psi\cdot\underline{\nabla} \right)p_0 = 0$ $\qquad$ (6.75)
k_0^0	$\left(\underline{\nabla}\cdot\nabla\psi + 2\underline{\nabla}\psi\cdot\underline{\nabla} \right)p_1 = -\nabla^2 p_0, \quad \dfrac{\nabla^2 p_1}{iR} = \nabla^2 p_0$ $\qquad$ (6.76)
$\vdots$	$\vdots$

Here, Eq. (6.74) is the <u>eiconal equation</u>, and Eqs. (6.75), (6.76) are the <u>transport equations</u>. The lowest order solution in Eq. (6.72) is the local plane-wave field

$$p \sim p_0\, e^{ik_0\psi(\underline{x})} \tag{6.77}$$

The acoustic intensity in this planewave field (dimensions: (N-m/sec)/m^2),

$$\underline{\bar{S}} = \zeta \,\mathcal{I}m\,(p^*\underline{\nabla}p), \quad \zeta = \text{const.} \tag{6.78}$$

$$\text{''} = \frac{1}{\varsigma_0 c_0}\, p_0^2\, \underline{\nabla}\psi(\underline{x}) \tag{6.79}$$

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The ray trajectories are curves parallel to the local time-averaged flux density vector $\overline{S}$. From these relations it is seen that Rytov's equation (in its various forms) is a "geometric acoustics" solution of the basic Helmholtz equation.

Parabolic Equation

We rewrite Eq. (6.71) for the case of a random index of refraction μ in the equivalent form

$$\left\{ \nabla^2 + k_0^2 \langle \mu \rangle \left[1 + \tilde{\mu}(\underline{x}) \right] \right\} p(\underline{x}) = 0 \qquad (6.80)$$

in which $\langle \mu \rangle$ is the mean value and $\tilde{\mu}$ is the (normalized) fluctuating component of μ. Let the scale of the inhomogeneities be "large" (i.e., let the wavelength be small relative to each scattering inhomogeneity). Now assume in (6.80) that

$$p(\underline{x}) = P(x) e^{ikx} \qquad , \qquad k^2 = k_0^2 \langle \mu \rangle \qquad (6.81)$$

Then, cancelling common terms in k^2 one arrives at,

$$2ik \frac{\partial P(\underline{x})}{\partial x} + \nabla^2 P(\underline{x}) = -k^2 \tilde{\mu} P(x) \qquad (6.82)$$

If the range x is much greater than the scale of inhomogeneity, and if the wavelength is much smaller than either, then

$$\frac{\partial^2 P}{\partial x^2} \ll 2k \left| \frac{\partial P}{\partial x} \right| \qquad (6.83)$$

and the full Laplacian is then replaced by the cross-sectional Laplacian $\nabla_{\perp}^2 = \partial^2/\partial y^2 + \partial^2/\partial z^2$. Then Eq. (6.82) reduces to the parabolic equation of propagation,

$$2ik \frac{\partial P}{\partial x} + \nabla_{\perp}^2 P = -k^2 \tilde{\mu} P \qquad (6.84)$$

The meaning of this equation is this: the local sound field interacts with the random inhomogeneity to constitute a source (= r.h.s.) which causes a slow diffusion (or scattering)

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of the energy in the sound field in the transverse direction as the wave propagates in the x-direction.

Conclusion of Sects. 4, 5 and 6

The mathematical description of wave motion in the ocean is very complex in the general case. Specific simplifications have been derived and their limits shown. In all cases the appropriate equations contain environmental parameters whose description require detailed investigation. A discussion of these parameters is undertaken in the next sections.

7. The Parameters of Ocean Inhomogenities and Their Statistical Measures

The parameters which describe ocean inhomogeneities are random variables. A statistical description of them is required to begin the formulation of a computational algorithm to allow prediction of the fluctuation and fading of acoustic signals. We present in the following sections an analysis of two types of ocean inhomogeneities and their associated statistical description. These types are: (a) surface and volume (velocity) turbulence (b) random index of refraction (or temperature inhomogeneities). The discussion of these types requires an understanding of the concept of scales of length and time. This is presented in the following sections.

8. Scales of Length and Time

We introduce the notion of scale by means of Fourier analysis. Let $\underline{u}(\underline{x},t)$ be a field (vector) variable, which is Fourier-transformable. The Four-dimensional Fourier transform $U(\underline{\kappa}, \omega)$ is (formally) given by the pair,

$$\underline{u}(\underline{x},t) = \iiiint \underline{U}(\underline{k},\omega) e^{i[\underline{\kappa} \cdot \underline{x} - \omega t]} \frac{d^3\underline{\kappa}\, d\omega}{(2\pi)^4} \qquad (8.1)$$

$$\underline{U}(\underline{\kappa},\omega) = \iiiint \underline{u}(\underline{x},t) e^{-i[\underline{\kappa} \cdot \underline{x} - \omega t]} d^3\underline{x}\, dt \qquad (8.2)$$

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Since wavenumber $|\mathcal{K}| = 2\pi \lambda^{-1}$, where λ = wavelength, Eq. 8.1 infers that the field at $\underline{x}$, t can be expanded (i.e., decomposed) into spatial component of different wavelengths, and temporal components of different frequencies. At a fixed frequency ω_0 the quantity $\underline{U}(\underline{\mathcal{K}}, \omega_0) d^3\underline{\mathcal{K}}$ is the spatial component (in density form) and at fixed wavenumber $\underline{\mathcal{K}}_0$, the quantity $\underline{U}(\underline{\mathcal{K}}_0, \omega) d\omega$ is the temporal component. Eq. 8.2 implies that at time $t = t_0$, the field $\underline{u}$ at $\underline{x}$, spread over a volume $d^3\underline{x}$, and effectively zero outside of this volume, corresponds to a wavelength λ such that $\lambda \in 2\pi |\mathcal{K}|^{-1}$. The volume of $d^3\underline{x}$ at $\underline{x}$ over which $\underline{u}(\underline{x}, t_0)$ effectively contributes is the volume scale of the spatial component of the field corresponding to the wavenumber $\underline{k}$ (or wavelength λ). The large scale (spatial) components of u(x,t) originate in the small wave numbers (Eq. 8.1). As $|\mathcal{K}|$ is increased the scale of the spatial components of $\underline{u}$ decrease. Thus Fourier analysis decomposes the spatial aspect of the field into components of different physical size. The relative magnitude of each component is given by $U(\underline{\mathcal{K}}, \omega)$.

We consider next that the field u is a random function of position and time. At a specified time the mean value of the product of $u(x_1, t_0)$ and $u(x_2, t_0)$)i.e., the auto-covariance) is used to define the (spatial) autocorrelation function, $\varsigma(x_1, x_2)$, written as a non-dimensional quantity, viz.

$$\langle \underline{u}(\underline{x}_1, t_0) \cdot \underline{u}^*(\underline{x}_2, t_0) \rangle = \langle u^2 \rangle \Big|_{\underline{x}_1 = \underline{x}_2} \varsigma(\underline{x}_1, \underline{x}_2) \tag{8.3}$$

For a homogeneous field,

$$\varsigma(\underline{x}_1, \underline{x}_2) = \varsigma(|\underline{x}_1 - \underline{x}_2|) = \varsigma(r) \tag{8.4}$$

Upon integrating the autocorrelation of a homogeneous field over all r one obtains the quantity J(dimensions: length)

$$J = \int_0^\infty \varsigma(r) \, dr \tag{8.5}$$

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Assuming this quantity exists (which requires ϱ (r) $\to$ 0 as r $\to\infty$), it can be used to define the integral scale of the random process $\underline{u}(\underline{x},t)$. Similarly, the curvature of the function ϱ(r) at the origin, viz., the quantity $d^2\varrho(r)/d r^2$, can be used to define the microscale (Λ) of the random process, namely,

$$\frac{d^2\varrho(r)}{dr^2}\bigg|_{r=0} \sim \frac{const.}{\Lambda^2} \tag{8.6}$$

In many physical processes, Λ is very small, leading to the appearance of a near cusp at the origin of ϱ (r).

Eq. 8.3 can also be used to define other scales. For example, the value of the distance r at which the autocorrelation is effectively negligible, is the autocorrelation scale of the random process $\underline{u}(\underline{x},t)$. This scale (L) might be the distance where ϱ (L) is 1/e of its value at ϱ (0), or 5% of ϱ (0), etc. Higher order (say > 2) moments of the random function $\underline{u}(\underline{x},t)$ also can serve to define scales.

The concept of scale has a more specific meaning in the theory of turbulence. We first consider the spectral kinetic energy density $E(\underline{\mathcal{K}})$ per unit volume in wave number space, i.e., the energy in the shell between $|\underline{\mathcal{K}}_1| = a$ and $|\underline{\mathcal{K}}_2| = b$,

$$E(a, b) = \int_a^b E(\underline{\mathcal{K}}) d^3\underline{\mathcal{K}} \tag{8.7}$$

For simplicity the random velocity field is taken to be spatially homogeneous and isotropic. The total kinetic energy T of this velocity field then may be written

$$T = \int_0^\infty E(\underline{\mathcal{K}}) d^3\underline{\mathcal{K}} \tag{8.8}$$

On the hypothesis that dissipation of this kinetic energy occurs in the range of the smallest scales (i.e., the largest wavenumbers), a convenient description of this rate of dissipation (written as ε) becomes,

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$$\mathcal{E} = 2\nu \int_0^\infty \mathcal{K}^2 E(\underline{\mathcal{K}}) d^3\underline{\mathcal{K}} \qquad ; \qquad \mathcal{K}^2 = |\underline{\mathcal{K}}|^2$$

$$(8.9)$$

(provided $E(\underline{\mathcal{K}})$ is such that the integral exists), in which ν is the kinematic viscosity (dimensions: m^2/sec). In the dissipation range the field is thus characterized by $\mathcal{E}$ and ν. By dimensional analysis one constructs a length $\hat{\ell}_0$, and a velocity V_0, such that

$$\ell_0 = \sqrt[4]{\frac{\nu^3}{\mathcal{E}}} \qquad ; \qquad V_0 = \sqrt[4]{\mathcal{E}\nu}$$

$$(8.10)$$

The length $\hat{\ell}_0$ is the inner (viz., smaller) scale of turbulence, and the velocity V_0 is the characteristic velocity of the dissipation range.

The energy of turbulence is generated in the energy range of wave numbers from sources such as winds, thermal gradients, Coriolis' effects, etc. Since the velocity field is random one chooses the mean value of flow generated by these sources, and defines from it a characteristic length L_O i.e. a length (in the energy range) over which the mean flow changes "appreciably." This length of the largest velocity disturbances is the outer scale of turbulence. Between the energy range (or cascade range) and the dissipation range there is a range of wave numbers (= the inertial range) over which (by the hypothesis of Taylor) the kinetic energy is transferred from smaller to larger wave numbers without loss (= lossless fragmentation of eddies). The mean velocity of flow diminishes, and with it the Reynolds number (= $V_{mean} L/\nu$) diminishes for each increase in wave number (and therefore each decrease in scale L) in the inertial range. For isotropic turbulence the inertial range and the dissipation range are (by hypothesis) statistically stationary. These two ranges constitute the equilibrium range of wave numbers and hence the equilibrium range of scales.

Orders of Magnitude in Turbulent Flow

The local instantaneous velocity $\underline{u}$ in turbulent flow is a sum of the local flow velocity $\underline{U}'$ and the local sound particle velocity $\underline{v}'$. (Here the mean velocity of flow $\underline{U}$ is assumed zero.) We let $\underline{U}$ be a reference rms turbulent velocity fluctuation. The integral scale L_u

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of turbulence (see Eq. 8.5) is rewritten here for component velocity $U_i{}'$ as,

$$L_u \equiv \int_0^\infty d\,|\underline{x}_1 - \underline{x}_2| < U_i{}'(\underline{x}_1)\, U_i{}'(\underline{x}_2) >,\ \text{meter}$$

(8.11)

(no sum on component $U_i{}'$), in which the correlation function is normalized to unity and is non-dimensional. From Eq. 8.6 one can define a (G. I. Taylor) microscale for velocity fluctuations (= Λ_g), and for temperature fluctuations (= Λ_θ)in terms of the turbulent velocity correlation functions. The squared ratio between them can be shown to be the Prandtl number P_r,

$$P_r = \frac{\Lambda_g^2}{\Lambda_\theta^2}$$

(8.12)

For water $P_r \simeq 7$ while for gases P_r is of the order of unity. Now in scaling turbulent phenomena it is useful to order the magnitudes of length by Λ_g and Λ_θ and time by use of the scale L_u . When sound is present one uses the sound speed c and the sound wavelength λ to order magnitudes. When the various scales are assembled the problem arises as to what criterion must one use to scale spatial and temporal changes. A convenient criterion is to refer to the Kolmogorov inertial subrange (namely the range in which kinetic energy is transferred from larger eddies to smaller eddies by the cascade process) and to note first that the most rapid time change of U' is at the upper bound of the sub-range where the wave number is approximately $1/L_u$. Thus the ratio U/L_u (with dimensions of time) serves to scale time, by which is meant that the smallest significant time interval describing the turbulent process is of the order of U/L_u. Choosing larger wave numbers (i.e., choosing smaller characteristic lengths) makes the time scale larger, with the result that only increasingly slower phenomena come under scrutiny. The scaling of length may similarly be referred to the inertial subrange where it is noted that the time associated with the cascade process at the microscale Λ_g of turbulent flow is U'/Λ_g . Hence the distance in the turbulent field over which a significant step in cascade process occurs is (at the minimum) of the order of Λ_g . Choosing distance smaller than the microscale Λ_g does not allow valid statistical deductions to be made of the turbulent

field. When sound waves are present a convenient time interval for time scaling is the ratio of the sound wave-length to sound speed (= λ/c). With these considerations in mind one may assume these orders of magnitude [7].

Time Scales

$$\frac{\partial}{\partial t} = \mathcal{O}\left(\frac{\mathcal{U}}{L_x}\right) \qquad \text{(for turbulent fluid and thermal fluctuations)}$$
$$(8.13)$$

$$\frac{\partial}{\partial t} = \mathcal{O}\left(\frac{c}{\lambda}\right) \qquad \text{(for sound waves)} \qquad (8.14)$$

Space Scales

$$\frac{\partial}{\partial x_i} = \mathcal{O}\left(\frac{1}{\Lambda_e}\right) \qquad \text{(for turbulent fluid)} \qquad (8.15)$$

$$\frac{\partial}{\partial x_i} = \mathcal{O}\left(\frac{1}{\Lambda_g}\right) \qquad \text{(for thermal fluctuations)} \qquad (8.16)$$

$$\frac{\partial}{\partial x_i} = \mathcal{O}\left(\frac{1}{\lambda}\right) \qquad \text{(for sound disturbances)} \qquad (8.17)$$

Additional scales involve forces and dissipation. For a fluid of density ρ , velocity U, the hydrodynamic stresses scale as $1/2(\rho U^2)$. Dissipation (or power) scales as $1/2(\rho U^3)$.

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Scaling of Vector Fields (Case of Isotropic Turbulence)

At a point $\underline{x}$ in the field $u(\underline{x})$ we designate a direction $\underline{r}$. Let the velocities at $\underline{x}$ and $\underline{x} + \underline{r}$ have components u_p (x), u_p ($\underline{x} + \underline{r}$) where u_p is the longitudinal component of velocity (i.e., parallel to $\underline{r}$) and u_n (x) $u_n(\underline{x} + \underline{r})$ where u_n is the perpendicular component of the velocity. There are then two scalar functions of the distance r, namely the longitudinal velocity correlation f(r), and the lateral velocity correlation g(r) given by the ensemble averages

$$f(\underline{r}) = \frac{\langle u_p(\underline{x}) \, u_p(\underline{x}+\underline{r}) \rangle}{\overline{u_p^2}} \quad ; \quad g(\underline{r}) = \frac{\langle u_n(\underline{x}) \, u_n(\underline{x}+\underline{r}) \rangle}{\overline{u_n^2}} \tag{8.18}$$

For isotropic turbulence let U^2 be the mean-square of any velocity component $U^2 = \overline{u_p^2} = \overline{u_n^2}$. The velocity correlation tensor R_{ij} (r) for the random velocity at two points separated by $\underline{r}$ is

$$R_{ij}(\underline{r}) = \langle u_i(\underline{x}) \, u_j(\underline{x}+\underline{r}) \rangle \quad , \quad i,j = 1, 2, 3 \tag{8.19}$$

If the random process is stationary one can construct a correlation spectral density $\Phi_{ij}(\underline{\varkappa})$ such that,

$$R_{ij}(\underline{r}) = \int \Phi_{ij}(\underline{\varkappa}) \, e^{i\,\underline{\varkappa}\cdot\underline{r}} \, d^3\underline{\varkappa} \tag{8.20}$$

If the random field is the isotropic correlation tensor has the special form

$$R_{ij}(\underline{r}) = U^2 \left(\frac{f-g}{r^2} r_i r_j + g \, \delta_{ij} \right) \tag{8.21}$$

in which r_i is the i'th component of the space vector $\underline{r}$. By definition f(0) = g (0) = 1. Due to isotropy f(r) can be expanded in even powers of r^2, i.e.

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$$f(\underline{r}) = 1 + \frac{1}{2} \left(\frac{\partial^2 f}{\partial r^2} \right)_{r=0} r^2 + \mathcal{O}(r^4) \tag{8.22}$$

Introducing the Taylor microscale Λ g by the relation

$$\left(\frac{\partial^2 f}{\partial r^2} \right)_{r=0} = - \frac{1}{\Lambda_g} \tag{8.23}$$

one approximates f, g near the origin r = 0 by the relations

$$f(r) \approx 1 - \frac{r^2}{2\Lambda_g^2} \quad ; \quad g(r) \approx 1 - \frac{r^2}{\Lambda_g^2} \tag{8.24}$$

Now the Navier-Stokes equation for an uncompressible fluid can be written at two points $(\underline{x}, x' = \underline{x} + \underline{r})$ whose velocities (in component form) are u_i, u_j respectively. Multiplying the first by u_j the second by u_i, then adding the two equations, and finally taking the ensemble average leads to the statement that

$$\frac{\partial R_{ij}}{\partial t} = T_{ij}(\underline{r}) + P_{ij}(\underline{r}) + 2\nu \nabla^2 R_{ij}(\underline{r}) \tag{8.25}$$

where

$$T_{ij} = \frac{\partial}{\partial r_k} \left(\langle u_i u_k u_j' \rangle - \langle u_i u_k' u_j' \rangle \right) \tag{8.26}$$

$$P_{ij} = \frac{1}{\varrho} \left(\frac{\partial \langle p u_j \rangle}{\partial r_i} - \frac{\partial \langle p' u_i \rangle}{\partial r_j} \right) \tag{8.27}$$

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Thus the temporal rate of change of the velocity correlation tensor is caused by the divergence of a third order velocity tensor describing inertia, plus the gradient of a hydrodynamic intensity which denotes the pressure effect, plus the Laplacian of the correlation tensor itself which describes the viscosity effect. Of the several possibilities in triple velocity correlations it is customary to select only one (for isotropic conditions) and define k(r) such that,

$$< u_p^2(\underline{x}) \, u_p \, (\underline{x}+\underline{r}) > = u^3 \, k(\underline{r})$$

(8.28)

With this choice the dynamical equation may be integrated with respect to r to obtain the v. Karman-Howarth equation,

$$\frac{\partial u^2 f(r)}{\partial t} = u^3 \left(\frac{\partial}{\partial r} + \frac{4}{r} \right) k(r) + 2\nu \, u^2 \left(\frac{\partial^2}{\partial r^2} + \frac{4}{r} \frac{\partial}{\partial r} \right) f(r)$$

(8.29)

Using the approximations for f(r), g(r) given by Eqs. (8.24) one performs the operations called for in this equation and then sets r = 0. The result is rate of decrease of energy of turbulence,

$$\frac{1}{u^2} \, \frac{du^2}{dt} = - \frac{10\nu}{\Lambda_g^2}$$

(8.30)

Using Eq. 8.13 for the temporal scale, one then arrives at the following formula for estimating turbulent magnitude in the case of isotropic turbulence,

$$\frac{10 \, \nu \, u^2}{\Lambda_g^2} = \vartheta \left(\frac{u^3}{L_u} \right)$$

(8.31)

or

$$\frac{\Lambda_g^2}{L_u^2} = \vartheta \left[\frac{10\nu}{\left(\frac{u}{c} \right) L_u c} \right]$$

(8.32)

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This formula can best be used to compare the Taylor microscales Λ_g of different media, provided the integral scales L_u, the viscosity ν, and the Mach number U/c are available.

Physical interpretation of the power spectrum of scalar fields

Let $\Theta(\underline{x}, t)$ be a random scalar (say temperature) field in a volume V over time duration T. For any two points $\underline{x}, t$ and $\underline{x} + \underline{d}$, $t + \tau$ we seek to measure the degree of self-similarity $\Upsilon(\underline{d}, \tau)$ of field, i.e., how well the field is coupled to itself over distance $\underline{d}$ and time τ. Integrating over all possible points separated by $\underline{d}$ and τ, one has

$$\Upsilon(\underline{d}, \tau) = \frac{1}{V\tau} \iint\limits_{V\tau} \Theta(\underline{x}, \tau)\, \Theta(\underline{x} + \underline{d}, t + \tau)\, d\underline{x}\, d\tau \tag{8.33}$$

The 4-dimensional plot of $\Upsilon(\underline{d}, \tau)$ vs $\underline{d}, \tau$ constitutes the autocorrelation of Θ. An integral of Υ over all space-time constitutes the integral space-time scale. The value of $\Upsilon(\underline{d}, \tau)$ can be expanded into an infinity of spatial and temporal frequencies, $\underline{K}$ ω,

$$\Upsilon(\underline{d}, \tau) = \iint\limits_{-\infty}^{\infty} W(\underline{K}, \omega)\, e^{-i\omega\tau + i\underline{K}\cdot\underline{d}}\, \frac{d\underline{K}\, d\omega}{(2\pi)^4} \tag{8.34}$$

Here $W(\underline{K}, \omega)$ is the 4-dimensional power spectrum of the random scalar field $\Theta(\underline{x}, t)$ For fixed time (= "frozen" medium) $W(\underline{K}, \omega)$ shows the distribution of plane wave intensities among various $\underline{K}$ which sum to give $\Upsilon(\underline{d}, t)$. Since $|\underline{K}|$ can be interpreted as $2\pi/L$, where L is a spatial scale associated with the scalar, temperature field it is seen that $W(\underline{K}, \omega)$ shows the distribution of plane wave intensities among

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different scales of length (= different wave-lengths at a fixed temporal frequency ω). The value of $W(\underline{K}, \omega)$ at a fixed scale $\underline{K}$ and frequency ω can be obtained as a sum of plane waves in space-time,

$$W(\underline{K}, \omega) = \iint \Gamma(\underline{d}, \tau) e^{i\omega\tau - i\underline{K}\cdot\underline{d}} d\tau d^3\underline{d} \qquad (8.35)$$

Hence the power of the scalar field in each scale can be obtained by scanning the field of plane waves of all τ and $\underline{d}$.

9. Statistical Measures of the Random Field I. Ref. [5]

In the propagation of scalar waves in a random medium several parameters may be used to describe the statistical nature of the propagation. A customary procedure is to begin with a specific wave form, say

$$u(\underline{x}, t) = A(\underline{x}) \exp\left\{ i S(\underline{x}) - i\omega t \right\} \qquad (9.1)$$

Here both the amplitude A and the phase S are random functions of position. We first suppose that there are an indefinite number of test spaces (i.e. samples of random media) and that we select two location vectors $\underline{x}_1$, $\underline{x}_2$ the same for each test space. The outcome of the experiment at each point is the sequence of random numbers,

$$A_1(\underline{x}_1) \, , \quad A_2(\underline{x}_1) \, \cdots \cdots \qquad A_\infty(\underline{x}_1)$$
$$S_1(\underline{x}_1) \, , \quad S_2(\underline{x}_1) \quad \cdots \cdots \qquad S_\infty(\underline{x}_1) \quad etc. \qquad (9.2)$$

or $\qquad A_i(x_j) \, , \quad S_i(x_j) \, , \qquad i,j = 1, 2, \cdots\cdots \infty$

in which the subscript i = 1, 2 . . . represents the test space in question. We then form the products

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$$\left. \begin{array}{c} A_i\left(x_j\right) A_i\left(x_k\right) \\[4mm] S_i\left(x_j\right) S_i\left(x_k\right) \end{array} \right] \quad \begin{array}{l} i = 1, 2, \cdots \cdots \infty \\[2mm] k = 1, 2, \cdots \cdots \infty \end{array} \qquad (9.3)$$

The ensemble average of $\quad A\left(\underline{x}_1\right) A\left(\underline{x}_2\right) \quad$ by definition is

$$\langle A(x_1) \cdot A(x_2) \rangle = \lim_{N \to \infty} \frac{A_1\left(\underline{x}_1\right) A_1\left(x_2\right) + A_2\left(\underline{x}_1\right) A_2\left(\underline{x}_2\right) + \cdots + A_N\left(\underline{x}_1\right) A_N\left(x_2\right)}{N}$$

$$(9.4)$$

A similar formula may be constructed for $\quad \langle S\left(\underline{x}_1\right) S\left(\underline{x}_2\right) \rangle \quad$. Many investigators use the log amplitude (= log A(x)) to describe the statistical nature of the field. If $U(\underline{x},t) \quad$ is nonharmonic the averaging process can be performed over t as well as $\underline{x}$.

In practical cases only one test space is available. A form of ergodic hypothesis may then be used to obtain the ensemble average. According to this hypothesis the random function (say A(x)) is measured over the single available test space and constitutes one realization. If an average is made (say mean value of A(x)), or mean value of $A\left(\underline{x}_1\right) A\left(\underline{x}_2\right)$ etc, then the hypothesis states that the variance of this average about the ensemble mean must vanish as the test space increases indefinitely. As an example we consider the mean value $\langle A\left(\underline{x}\right) \rangle$ in the test space. Then the hypothesis states that the variance σ^2 of $A(\underline{x})$ over the volume V of the test space must obey the relation,

$$\lim_{V \to \infty} \left(\sigma^2(\underline{x}) = \left\langle \frac{1}{V} \left[\int_0^V A(\underline{x} + \underline{x}')d\underline{x}' - \langle A(\underline{x}) \rangle \right]^2 \right\rangle \right) \to 0$$

$$(9.5)$$

that is, the square of the difference between the space average of $A(\underline{x})$ with $\underline{x}$ as origin over the volume V, and the true ensemble mean of $A(\underline{x})$ is a random function whose ensemble variance about the ensemble mean approaches zero as the volume increases. Noting that in all practical cases the space average is the only measurable quantity it is seen that the ensemble mean at $\underline{x}$ is obtainable by estimating the volume integral appearing in Eq. (9.5). In one dimension, for a length L, the hypothesis takes the form

$$\lim_{L \to \infty} \left(\sigma^2(x) = \left\langle \frac{1}{L} \left[\int_0^L A(x+x') \, dx' - \langle A(x) \rangle \right]^2 \right\rangle \right) \to 0$$

(9.6)

In order for this limit to hold the length of record L must be long enough to contain all integral scales (defined in Eq. (8.5). The question naturally arises as to whether the length of record is adequate in this respect. Following Lumley-Panofsky [9] we discuss the problem in the following way. Let the ensemble variance of A(x) about its ensemble mean be $\langle f'^2 \rangle$, that is, let

$$\langle f'^2 \rangle \equiv \left\langle \left[A(x) - \langle A(x) \rangle \right]^2 \right\rangle$$

(9.7)

Now choose an acceptable level of error ϵ in the value of σ as expressed in Eq. 4.5, by writing

$$\sigma = \langle A(x) \rangle \, \epsilon$$

(9.8)

Then, if the integral scale J of the correlation function for the random field A(x) is known, the length L is approximately

$$L \cong 2 \, \frac{\langle f'^2 \rangle}{\langle A(x) \rangle^2} \, \frac{J}{\epsilon^2}$$

(9.9)

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Thus in order to determine the ensemble mean of any moment of the field from a single (finite sized) test space one requires a knowledge of the integral scale J and, upon consideration, of all integral scales of the random process. Ordinarily the relation between moments is not known so that a single J is inadequate. To overcome lack of knowledge a convenient strategy is to assume that the random field in question has a gaussian distribution. Then one measures the autocorrelation of the process. Since the relation between moments in a gaussian process is known, a size of sample space for each moment can then be determined using only a single integral scale.

We next turn to the second moments of the random field of A(x), i.e. $\langle A(\underline{x}_1)$ $\times A(\underline{x}_2)\rangle$. If only one space realization is available and if there is no restriction on the statistical nature of the field then no ensemble average by use of the ergodic hypothesis is possible for a specific set of points. However, if the random field is statistically homogeneous then the second moment depends only on separation distance between two space points but not on the points themselves. In this case a single realization can be used (by the ergodic hypothesis) to represent the total random process. The ergodic hypothesis then requires

$$\lim_{V \to \infty} \left(\sigma^2 = \langle \frac{1}{V} \left[\int A(x+\Delta x) A(x)\, dx - \langle A(x+\Delta x)A(x)\rangle \right]^2 \rangle \right) \to 0$$

(9.10)

Hence the ensemble average (or second moment) is approximated by the space average $\mathcal{R}(\Delta x)$,

$$\mathcal{R}(\Delta x) = \frac{1}{N} \left[A(\underline{x}_1 + \Delta \underline{x}) A(\underline{x}_1) + A(\underline{x}_2 + \Delta \underline{x}) A(\underline{x}_2) \cdots + A(\underline{x}_N + \Delta \underline{x}) A(\underline{x}_N) \right]$$

(9.11)

Now for an ideal homogeneous random process the size V of the sample space can be increased indefinitely making σ^2 smaller and smaller. For a realistic process it generally happens that the increase of V beyond a critical amount causes σ^2 to increase again. In these cases the onset of the departure from homogeneity in a volume V corresponds to a residual error (i.e. there is a minimum σ^2). Increasing V beyond this critical V_{crit} only serves to increase the residual error.

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10. Statistical measures of the random field II

A random field expressed as a real function of space and time $p(\underline{x}, t)$ possess a space-time correlation function Υ between two points in space-time,

$$\Upsilon(\underline{x}_1, x_2; t_1, t_2) = \langle p(\underline{x}_1, t) \, p(\underline{x}_2, t) \rangle \tag{10.1}$$

in which the sharp brackets $\langle \rangle$ indicate ensemble averages over space-time. Using $\underline{d} = (\underline{x}_2 - \underline{x}_1)$, $\tau = t_2 - t_1$, one defines a space-time autocorrelation function in the case of a homogeneous stationary medium by the relation

$$\Upsilon(\underline{d}, \tau) = \langle p(\underline{x}_1, t_1) \, p(\underline{x}_1 + \underline{d}, t + \tau) \rangle \tag{10.2}$$

When only one realization is available the space-time autocorrelation can be calculated by invoking an ergodic hypothesis, i.e.

$$\Upsilon(\underline{d}, \tau) = \frac{1}{VT} \int\int p(\underline{x}, t) \, p(\underline{x} + \underline{d}, t + \tau) \, dt \, d\tau \tag{10.3}$$

in which V is a sample volume and T a sample time duration.

When $p(\underline{x}, t)$ is defined for all values of $\underline{x}$, and t it possess a Fourier representation

$$p(\underline{x}, t) = \int \Big[a_{\nu, k} \cos(-\underline{K} \cdot \underline{x} + 2\pi \nu t) + b_{\nu, k} \sin(-\underline{K} \cdot \underline{x} + 2\pi \nu t) \Big] d^3\underline{x} \, d\nu \tag{10.4}$$

in which $a_{\nu, \kappa}$, $b_{\nu, \kappa}$ are real Fourier coefficients in ν, K space. With $p(\underline{x}, t)$ one associates a complex function $\hat{p}(\underline{x}, t)$ given by

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$$\hat{p}(\underline{x}, t) = 2 \iiint\limits_{0}^{\infty} (a_{\nu,\kappa} + i b_{\nu,\kappa}) e^{-2\pi i \left[-\underline{\kappa} \cdot \underline{x} + \nu t \right]} d^3\underline{\kappa} \, d\nu$$

(10.5)

$$p(\underline{x}, t) = \text{Real part of} \quad \hat{p}(\underline{x}, t)$$

This symbol $\hat{p}(\underline{x}, t)$ is the half-range complex function associated with $p(\underline{x}, t)$. It is used to define the mutual intensity function of Zernike ($= \Gamma$), where

$$\hat{\Gamma}(\underline{x}_1, \underline{x}_2, \tau) = \left\{ \hat{p}(\underline{x}_1, t + \tau) \hat{p}^*(\underline{x}, t) \right\}$$

The curly brackets indicate <u>time</u> average. Since $\hat{p}$ is random in space, $\hat{\Gamma}$ is random. The <u>ensemble</u> averaged $\hat{\Gamma}$ is

$$\langle \hat{\Gamma}(\underline{x}_1, \underline{x}_2, \tau) \rangle = \left\langle \left\{ \hat{p}(\underline{x}_1, t + \tau) \hat{p}^*(\underline{x}_2, \tau) \right\} \right\rangle$$

(10.6)

If $p(\underline{x}, t)$ is defined over the range $-T \le t \le T$, and vanishes outside, then its Fourier components in time are

$$p(\underline{x}, \nu) = \int_{-T}^{T} p(\underline{x}, t) e^{2\pi i \nu t} dt$$

(10.7)

Therefore,

$$\hat{\Gamma}(\underline{x}_1, \underline{x}_2; \tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{0}^{\infty} p(\underline{x}_1, \nu) p^*(\underline{x}_2, \nu) e^{-2\pi i \nu \tau} d\nu$$

(10.8)

The complex degree of mutual coherence ν is defined as the normalized $\hat{\Gamma}$.

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$$\langle \hat{\gamma} (\underline{x}_1, \underline{x}_2, \tau) \rangle = \frac{\langle \hat{\Gamma} (\underline{x}_1, \underline{x}_2, \tau) \rangle}{\sqrt{\hat{\Gamma} (\underline{x}_1, \underline{x}_1; 0)} \sqrt{\hat{\Gamma} (\underline{x}_2, \underline{x}_2; 0)}} \tag{10.9}$$

(Compare Eq. 1.6)

When $\tau = 0$ the average power Wav of the field $\not{p} (\underline{x}, t)$ received by an array of N points not phased relative to each other is

$$W_{AV} = \frac{1}{2} \sum_{k=1}^{N} \hat{\Gamma} (X_k, X_k; 0) \tag{10.10}$$

The average power Wav received by a single element of this array is

$$W_{AV} = \frac{1}{2N} \sum_{k=1}^{N} \hat{\Gamma} (X_k, X_k; 0) \tag{10.11}$$

Thus the array gain G is

$$G = \frac{\sum_{k, \ell=1}^{N} \hat{\Gamma} (X_k, X_L; 0)}{\frac{1}{N} \sum_{k=1}^{N} \hat{\Gamma} (X_k, X_k; 0)} \tag{10.12}$$

(Compare Eq. 1.7)

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11. Statistical measures of the random field III

Two nonstationary random processes $p(t)$, $u(t)$, $0 \le t < \infty$, with mean values $\langle p(t) \rangle$, $\langle u(t) \rangle$, defined to be the ensemble averages, (i.e. angle brackets) at arbitrary time t, can be used to form the covariance functions C at arbitrary fixed values of times t_1, t_2, i.e.

$$C_p (t_1, t_2) = \langle [p(t_1) - \langle p(t_1) \rangle][p(t_2) - \langle p(t_2) \rangle] \rangle$$

$$C_u (t_1, t_2) = \langle [u(t_1) - \langle u(t_1) \rangle][u(t_2) - \langle u(t_2) \rangle] \rangle \tag{11.1}$$

$$C_{pu} (t_1, t_2) = \langle [p(t_1) - \langle p(t_1) \rangle][u(t_2) - \langle u(t_2) \rangle] \rangle$$

The correlation function R between p and u at fixed times t_1, t_2 are defined as

$$R_p (t_1, t_2) = \langle p(t_1) p(t_2) \rangle$$

$$R_u (t_1, t_2) = \langle u(t_1) u(t_2) \rangle \tag{11.2}$$

$$R_{pu} (t_1, t_2) = \langle p(t_1) u(t_2) \rangle$$

The symbols C, R are statistical measures of the total population. Measures based on N sample functions $p_i(t)$, i = 1, 2, . . . N drawn from the total population are designated $\hat{C}$, $\hat{R}$, where for example,

$$\hat{R}_p (t, t - \tau) = \frac{1}{N} \sum_{i=1}^{N} p_i(t) p_i(t - \tau) \tag{11.3}$$

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Time varying power spectra (Bendat & Piersol, [10] p. 337).

From a population x (t) of non-stationary random processes we draw a sample $x_i(t)$ and calculate the power (i.e. $x_i^2(t)$) in a frequency band $\mathcal{B}_e$ cycles wide centered at frequency f. A filter with weighting function h (τ) is applied to filter out this band from x_i (t), which is then squared.

The result is

$$x_i^2 (t; f, \mathcal{B}_e) = \iint_0^\infty h(\xi) \, h(\eta) \, x_i (t-\xi) x_i (t-\eta) \, d\xi d\eta$$

(11.4)

(Note, energy is considered only in positive f. The spectrum is then only one sided). By use of N samples x_i (t), i = 1, 2 .. N, an estimate of the ensemble average of $x_i^2 / \mathcal{B}_e$ can be made for arbitrary fixed time t,

$$\hat{G}_x (t, f) = \frac{\langle x_i^2(t) \rangle}{\mathcal{B}_e} = \frac{1}{N \mathcal{B}_e} \sum_{i=1}^N x_i^2 (t; f, \mathcal{B}_e)$$

(11.5)

This is the one-sided time varying spectrum for fixed t, f. Since the estimation of transients are the object of forming this spectrum it is necessary that the bandwidth of the filter be greater than the highest frequency F_t of the transients in the data of x_i (t).

An estimate of the time-averaged power spectrum can be made by integrating $\hat{G}(t, f)$ over a time T,

$$\left\{ \hat{G} (t, T) \right\} = \lim_{T \to \infty} \frac{1}{T} \int_T \hat{G}_x (t, f) \, dt$$

(11.6)

An adequate theory of time varying power spectra is difficult to pursue in detail. A basis for understanding this theory is found in the above noted reference.

11.2 Statistical Model of a Mulitpath Signal (Dyson et. al, JASA 59 1121 (1976))

An acoustic source at single frequency ω_0 radiates to a distant receiver. The observed pressure is modeled as a superposition of single path components, $i = 1, 2 ...M$ (see Eq. 1.3f). Because of (1) fluctuations in the medium, (2) scattering from volume surface and bottom, (3) range-dependent acoustic velocity profiles, and (4) other factors such as tides, currents etc. the measured signal in each single-path appears as a narrowband noise process $Z_i(t)$, modeled as

$$Z_i(t) = X_i(t) \cos \omega_0 t + Y_i(t) \sin \omega_0 t = R_i(t) \cos \left[\omega_0 t - \phi_i(t) \right]$$

(11.2.1)

in which X, Y are slowly varying random functions of t, and $R_i(t)$, $\phi_i(t)$ are their polar forms. The multipath signal Z (t) is a sum of all single-paths, the orthogonal components of which are,

$$X(t) = \sum_{i=1}^{m} R_i \cos \phi_i \quad , \qquad Y(t) = \sum_{i=1}^{n} R_i \sin \phi_i$$

(11.2.2)

A basic model often used by analysts assumes $R_i(t)$ and $\phi_i(t)$ to be <u>independent</u> random variables. This assumption is used below.

To form a statistical description of the multipath signal we first form the covariance function K of each Cartesian component, i.e.

$$K = \langle X(t) X(t+\tau) \rangle + \langle Y(t) Y(t+\tau) \rangle$$

$$'' = \sum_{i=1}^{m} \langle R_i^2 \rangle_i \ \langle \cos \phi(t) \cos \phi(t+\tau) + \sin \phi(t) \sin \phi(t+\tau) \rangle$$

(11.2.3)

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in which the brackets $\langle \ \rangle_i$ indicate averaging over individual paths, and $\langle \ \rangle$ indicate averaging over the sum of all paths simultaneously. We assume next that the statistical average over the phase angles is the same for each phase difference so that

$$K = \langle \mathcal{R}e \ e^{i[\phi(t) - \phi(t+\tau)]} \rangle \sum_{i=1}^{n} \langle R_i^2 \rangle_i \tag{11.2.4}$$

Now the phase difference [$\phi(t) - \phi(t+\tau)$] is a random variable. We assume it to be normally distributed. We then expand it in a Taylor series and write,

$$\phi(t) - \phi(t+\tau) = \phi_o(t) + \tau \dot{\phi}(t) + \cdots \tag{11.2.5}$$

and seek an explicit expression for $\langle \mathcal{R}e \ \exp(\tau \dot{\phi}(t)) \rangle$ neglecting $\phi_o(t)$ because we are interested here in <u>phase differences</u>.

The expectation of $\exp(\tau \dot{\phi}(t))$ is recognized as the <u>characteristic function</u> of the random variable $\dot{\phi}(t)$,

$$\langle \exp(\tau \dot{\phi}(t)) \rangle = \int_{-\infty}^{\infty} w(t) \ e^{i\tau \dot{\phi}(t)} \, dt \tag{11.2.6}$$

where $w_{\dot{\phi}}(t)$ is the probability density of $\dot{\phi}(t)$. Assuming $\dot{\phi}(t)$ is a zero-mean Gaussian random variable one arrives at the result that

$$\langle \exp(\tau \dot{\phi}(t)) \rangle = \exp(-\sigma_{\dot{\phi}}^2 \tau^2/2) \tag{11.2.7}$$

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in which $\sigma_{\dot\phi}^2$ is the variance of $\dot\phi$ (t) (see Middleton "Statistical Information Theory" page 336). By noting that the covariance of X and the covariance of Y are equal it is then seen that

$$\langle X(t) \, X(t+\tau) \rangle = \frac{1}{2} \, exp\left(-\frac{1}{2} \dot\phi^2 \tau^2\right) \sum_{i=1}^{n} \langle R_i^2 \rangle_i. \quad (11.2.8)$$

The (power) spectrum (= F_X) of X(t) and of Y(t) are obtained from this formula by Fourier transformation;

$$\begin{cases} F_X(\omega) \\ F_y(\omega) \end{cases} = \frac{1}{\sqrt{2\pi \sigma_{\dot\phi}^2}} \left(\sum_{i=1}^{n} \langle R_i^2 \rangle_i\right) exp\left(-\frac{1}{2} \frac{\omega^2}{\sigma_{\dot\phi}^2}\right) \quad (11.2.9)$$

This spectrum is the Cartesian statistical model adopted by Dyson et al (JASA. $\underline{59}$ 1121 (1976)). In their model they give special prominence to $\sigma_{\dot\phi}^2 = \langle \dot\phi_i^2 \rangle_i$, which they designate as ν^2 (= frequency squared). The numerical calculation of ν^2 from experimental data of X, X, Y, Y is accomplished by using the relation

$$\nu^2 = \frac{\langle \dot X^2 \rangle + \langle \dot Y^2 \rangle}{\langle X^2 \rangle + \langle Y^2 \rangle}$$

The parameter ν^{-1} (= $\sqrt{\sigma_{\dot\phi}^2}$) when so obtained constitutes an experimentally

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measured time scale which scales the frequency ω of the spectrum. Now the intensity of the fluctuating signal is proportional to $R^2 = X^2 + Y^2$, where X, Y are zero mean Gaussian random variables. The statistics of fluctuations in intensity of the received signal in the ocean are most appropriately modeled in terms of $y = \ln R^2$ rather than R^2 itself since transmission loss is commonly a log quantity. To obtain the statistics of y we first note that R^2 obeys chi-square statistics with two degrees of freedom (namely X, Y), whose probability density is

$$ w(R^2) = \frac{1}{2\sigma_{R^2}^2} \; e^{-\frac{R^2}{2\sigma_R^2}}, \qquad R^2 \geqslant 0 \tag{11.2.10} $$

(Papoulis "Probability, etc." page 250). If now we change the significant variable from R^2 to $y = \ln R^2$, $w(y) = w(R^2)(dR^2/dy) = W(R^2)\,e^y$, it is seen that

$$ w(y) = \frac{1}{2\sigma_{R^2}^2} \; \exp\left[-\frac{e^y}{2\sigma_{R^2}^2} + y \right] \tag{11.2.11} $$

The variance (= $\langle y^2 \rangle - \langle y \rangle^2$) of this distribution is calculated by integration and has the value $\sigma_y^2 = \pi/6$ (Cramer, "Methods of Mathematical Statistics" p. 376). Since the transmission loss of the received <u>intensity</u> (= $\langle p^2 \rangle / p_o^2$, p_o^2 = source intensity) is a fluctuating quantity we desire to find its standard deviation. As noted above it is customary to report this quantity in logarithmic units (to the base 10 re 1 meter). At one standard deviation the fluctuation in intensity, I_{RMS} in decibels is $I_{RMS} =$ 4.34 $\sqrt{\sigma_y^2}$ (the constant 4.34 = 10/ln 10, accounts for the difference between log and ln). Hence $I_{RMS} = 5.57$ dB. This number depends only on the probability model used. The mean value of the transmission loss on the other hand depends on the transmission channel characteristics, range, attenuation, etc. (see Horton, Urick, etc. for pertinent formulas).

The rate of change of the multipath log intensity $|\dot{y}|$ is correlated to the rate of change of the multipath phase $|\dot{\phi}|$. Assuming $\dot{\phi}(t)$ is uniformly distributed in 0, 2π, and assuming the probability distribution of y is modeled as shown above it can be shown that the correlation between them is

$$C = \frac{\langle |\dot{y}| |\dot{\phi}| \rangle}{\langle \langle \dot{y}^2 \rangle \langle \dot{\phi}^2 \rangle \rangle^{\frac{1}{2}}} = \frac{2}{\pi} = 0.63 \qquad (11.2.12)$$

(Dyson et al, loc. cit.)

The dominant behaviour of intensity and phase of the received signal y and ϕ is closely associated with the occurrence of fade-outs. At any instant the intensity is proportional to $\sum \langle R_i^2 \rangle_i$. Writing this as μ^2 one defines a fade-out as a time interval in which $R < \epsilon \mu$, where ϵ is a threshold fraction. A fade-out expressed in dB is written as $F = 20 \log_{10} \epsilon^{-1}$. Assuming (as before) that X, Y are independent Gaussian random variables, and that R is Rayleigh distributed one can form a rough statistical theory of fade-outs (for the method see "Noise and Stochastic Processes," Ed. N. Wax, Dover Publications, N.Y. page 52ff). The (Dyson et al) fade-out model predicts (1) the average duration of the fade-out is $\tau = \frac{1}{2} \pi^{3/2} \epsilon \nu^{-1}$ (2) the average interval between fade-outs is $T = 1/2 \pi^{3/2} (\epsilon \nu)^{-1}$, (3) for chosen ϵ the fraction of time (of a record) occupied by fade-outs is ϵ^2. Using this model of fade-outs and a random walk model of X, Y, it is possible to predict the spectra of y and ϕ. At high frequencies (namely $\omega \gg \nu$) the model yields $F_\phi(\omega) = \frac{1}{2} F_y(\omega) = \frac{1}{2} \nu^2 \omega^{-3}$. At low frequencies, ($\omega < \nu$), $F_\phi(\omega) = (a\nu/\pi) \omega^{-2}$ where a = 1.74, and $F_i(\omega)$ is determinable from the fact that $\int_0^\infty F_i(\omega) d\omega = \pi^2/6$. These regimes of high and low frequency must be combined. After numerical experimentation Dyson et al constructed the following prediction model of the spectra of intensity and phase of the received signal,

$$F_\phi(\omega) = \nu^2 \omega^2 (\omega^2 + 1.27 \nu^2)^{-\frac{1}{2}} \qquad (11.2.13)$$

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$$F_y(\omega) = 4\nu^2 \left(\omega^2 + 2.43\,\nu^2\right)^{-3/2} \qquad (11.2.14)$$

Comparison with observed data of the project MIMI experiments (Section 2.0 of this report) shows that the random walk model predicts the high frequency behaviour over some ranges fairly well, but not over other ranges. In contrast, all comparisons of the observed very low frequency spectra of y and ϕ differ considerably from the random walk model. To understand this it is speculated that large scale ocean features are <u>coherently</u> modulating the signal so that a random walk model is inadequate in the very low frequency region.

The random walk model of Dyson et al is closely associated with the theory of strong scattering. An account of this theory is given in Section 31 of this report. Here it is shown (see Eq. 31.27) that the received signal is a sum of a weakened central field which is log-normally distributed, and an off-axis field which is Rayleigh distributed, the relative proportions being accounted for by multiple scattering. In the case of weak scattering the probability distribution of the received field is approximately log normal, the off-axis field being negligible. In the case of strong scattering the forward scattering component becomes negligible, and all the energy goes into the off-axis or Rayleigh component. Thus in application to the propagation of an initially monochromatic signal with well defined wavefront the model of Dyson et al becomes valid only at ranges where the Rayleigh distributed (or random walk) component of the received signal dominates the coherent component. At ranges and frequencies where coherence phenomena dominate the received signal the model does not predict the spectra of the received intensity and phase correctly. One must apply instead the theory of weak scattering and coherent interference.

12. Mathematical Model of the Index of Refraction as a Random Process

In physical media (ocean, air) the index of refraction is a random variable, mostly continuous. Of the many classes of random processes it will be convenient to list below the few that are of importance in volume scattering of scalar waves.

Let $X(t)$, $t_0 \leqslant t \leqslant t_5$ be a random process whose complete specification is given by <u>all</u> the joint probability densities

$$\mathcal{P}\left[X(t_i), \underline{B}\right], \quad B = \left\{X(t_{i+1}), X(t_{i+2}), \cdots X(t_{i-1}), \cdots\right\} \tag{12.1}$$

$i = 1, 2, 3 \ldots$ If the set $\underline{B}$ is restricted to one member, such that

$$\mathcal{P}\left[X(t_i) \mid \underline{B}\right] = \mathcal{P}\left[X(t_i) \mid X(t_{i+1})\right] \tag{12.2}$$

then $X(t)$ is a markov process. If, in addition, $p\,[X(ti)]$ and $p\left[X(t_i) \mid X(t_{i+1})\right]$ are both gaussian probability densities, then $X(t)$ is a gauss-Markov random process. If, for all t,

$$\mathcal{P}\left[X(t_i) \mid X(t_{i+1})\right] = \mathcal{P}\left[X(t_i)\right]$$

then $X(t)$ is a white noise process. If, in addition to the white noise property the probability density $p\,[X(t)]$ is gaussian, then $X(t)$ is a gaussian white noise process, usually designated as $w(t)$. A zero-mean gaussian white noise process is specified by the properties,

$$\mathcal{E}\left\{w(t)\right\} = 0 \;; \quad \mathcal{E}\left\{w(t), w(\tau)\right\} = \Psi_W(t)\,\delta(t-\tau) \tag{12.3}$$

in which Ψ_W is the variance of w at t. Gaussian white noise is not mean-square integrable.

Now let $F_T(t)$ and $f(t)$ be two random processes such that for fixed T, the difference

$$F_T(t) = f(t+T) - f(t) \tag{12.4}$$

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is a stationary random process. Then f (t) defines a non-stationary random function having stationary first increments, with initial value f (o). For random processes in volume scattering an important example of f (t) is the Wiener-Levy process $\beta(t)$, defined by the properties: (1) $\beta(t)$ has stationary first increments, all statistically independent; (2) $\beta(t)$ is gaussian distributed at any t ; (3) $\mathcal{E}\{\beta(t)\} = 0$ for all t; (4) $\beta(0) = 0$. From property (1) it can be concluded that

$$ VAR \left[\beta(t_1) - \beta(t_2) \right] = \sigma^2 (t_1 - t_2) \tag{12.5} $$

From property (3) it can further be shown that the covariance C_β has the form,

$$ C_\beta (t_1, t_2) \equiv \mathcal{E} \left\{ \beta(t_1) \, \beta(t_2) \right\} = \begin{cases} \sigma^2 t_1, & t_1 \leq t_2 \\ \sigma^2 t_2, & t_1 > t_2 \end{cases} \tag{12.6} $$

Noting that $C_\beta (t_1, t_2)$ is continuous, it is inferred [Jazwinski, p. 61] that β (t) is mean square continuous and mean-square (Riemann) integrable. However the covariance of the first derivative of β (t), namely the quantity,

$$ C_{\dot\beta} (t_1, t_2) \equiv \frac{\partial^2 C_\beta}{\partial t_1 \, \partial t_2} \tag{12.7} $$

is nowhere mean-square differentiable. Using Eq. (12.5), and the theory of generalized functions, it can further be shown that

$$ C_{\dot\beta} (t_1, t_2) = \sigma^2 \, \delta(t_1 - t_2) \tag{12.8} $$

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(Jazwinski,* p. 85). Thus while the covariance function of the Wiener-Levy process is continuous, the covariance of its <u>first derivative</u> is a delta-correlated process (i.e. formally, $\dot{\beta}(t)$ is white gaussian noise).

<u>Stochastic Differential Equation With Gaussian White Noise Forcing Function</u>

Let $\underline{X}(t)$ be a vector random process whose first derivative satisfies a Langevin-type dynamic equation driven by white noise,

$$\dot{\underline{X}}(t) = \underline{f}\left[\underline{X}(t), t\right] + \underline{\underline{G}}\left[\underline{X}(t), t\right]\underline{w}(t)$$

(12.9)

in which $\underline{X}$ is an n-dimensional state vector, $\underline{w}(t)$ an m-dimensional vector random disturbance in the form of additive gaussian white noise, $\underline{\underline{G}}$ is an $n \times m$ matrix function, and $\underline{f}$ is a nonlinear, real n-vector function. Since $\underline{w}(t)$ is not mean-square integrable this equation is mathematically undefined. To give it meaning one notes that (formally) $\underline{w} = d\underline{\beta}/dt$ is the first (scalar) derivative of a random Brownian motion process ($= \underline{\beta}$). Thus, setting $\underline{w} = d\underline{\beta}/dt$, inserting this into Eq. (12.9), and then integrating, one arrives at the form,

$$\underline{X}(t_1) - \underline{X}(t_2) = \int_{t_1}^{t_2} \underline{f}\left[\underline{X}(\tau), \tau\right] d\tau + \int_{t_1}^{t_2} \underline{\underline{G}}\left[\underline{X}(\tau), \tau\right] d\underline{\beta}(\tau)$$

(12.10)

Here the second integral on the r.h.s. is given a meaning by specifying it to be an <u>Itô</u> <u>stochastic integral</u>. Eq. (12.10) can be interpreted as a differential equation in which the integrals are replaced by finite increments; that is, it can be written as

$$d\underline{X}(t) = \underline{f}\left[\underline{X}(t), t\right] dt + \underline{\underline{G}}\left[\underline{X}(t), t\right] d\underline{\beta}$$

(12.11)

This is a stochastic differential equation in the random variable X(t), called an <u>Itô</u> <u>process</u> relative to (vector) matrices f and $\underline{\underline{G}}$. Eq. (12.11) defines a vector Markov process,

*[Ref. 18]

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i.e. it defines a filter in which the input is white noise (multiplied by $\underline{G}$) plus $\underline{X}(t)$, and the output is $\underline{\dot{X}}(t)$.

The vector $\underline{X}(t)$ in the theory of volume scattering has several components, one of which is the random index of refraction. The modeling of this quantity as a random process generated by an $\hat{I}to$ differential equation is discussed in the next section.

Analytical Modeling of the Random Index of Refraction

The index of refraction $\mu(\underline{x})$ in one dimension can be modeled as a colored noise process, i.e. as a (statistically) homogeneous, exponentially correlated Gaussian process. Its specifications are,

$$\mathcal{E}\left\{ \mu(\underline{x}) \right\} = 0$$

(12.12)

$$\mathcal{E}\left\{ \mu(\underline{x}+\underline{d})\,\mu(\underline{x}) \right\} = \frac{\sigma^2}{2D}\, e^{-\frac{|\underline{d}|}{D}} = \text{cov}\{\mu\}$$

in which D is (essentially) the correlation distance. As $D \rightarrow 0$, the limit $\text{cov}\{\mu\} \rightarrow \sigma^2 \delta(|\underline{d}|)$. Thus in the limit of zero correlation distance the set $\left\{ \mu(\underline{x}) \right\}$ formally defines a white gaussian process (i.e. a delta correlated random variable). If the process $\mu(\underline{y})$ is itself not a white noise process i.e. if Eq. (12.12) is restricted to finite D, then a physical process y (x), defined by

$$\frac{dy}{dx} = f\left[\, y(\underline{x}), \underline{x}\,\right] + g\left[\, y(\underline{x}), \underline{x}\,\right]\mu(\underline{x})$$

(12.13)

is not a Markov process. However, y (x) as defined by (Eq. 12.13) is smooth, differentiable and well defined.

The modeling of the index of refraction as a colored noise process has the advantage that it can be generated by a linear stochastic differential equation forced by white noise, namely

$$d\mu(x) = -\frac{\mu}{D}\, dx + \frac{\sigma}{D}\, d\beta(x)$$

(12.14)

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in which, as before, $d\beta$ is the differential of Brownian motion (Jazwinski, p. 123). In this special form the random variable μ is a Gauss-Markov process. Although the random process $y(x)$ is non-Markovian and the process $\mu(x)$ is Markovian, the vector $\underline{X}$ formed by $\underset{\sim}{y}$ and μ is a Markov process [Jazwinski, p. 105]. This is an example of state variable augmentation. The technique of augmentation is described in the next section.

State Vector Augmentation

Let the set $\underline{X}(k)$, k=0, 1, 2 . . . . be a random sequence of scalar or vector variables with functional dependence on a single parameter k. Any relation between $X(k)$ and $X(k+1)$, together with additive white noise ($= w(k)$), will generate a Markov sequence. For example the scalar sequence,

$$X_1(k+1) = C(k) X_1(k) + w(k) \qquad (12.15)$$

is a Markov sequence since it relates X_1 between two successive points, k, k +1, driven by white noise. Next suppose there are functions $C_1(k)$, $C_2(k)$ which relate X_1 between three successive points,

$$X_1(k+1) = C_1(k) X_1(k) + C_2(k) X_1(k-1) + w(k)$$
$$(12.16)$$

Here X_1 does not define a Markov sequence. It can be made into a Markov sequence by augmentation of the state scalar into a state vector with two components. This is done by defining the second component X_2 as $X_2(k) = X_1(k-1)$. Thus, the equations of state become

$$\underline{X}(k+1) = \underline{\underline{C}}(k) \underline{X}_1(k) + \underline{w}(k) \qquad (12.17)$$

in which

$$\underline{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad ; \quad \underline{\underline{C}}(k) = \begin{bmatrix} C_1(k) & C_2(k) \\ 1 & 0 \end{bmatrix} \quad ; \quad \underline{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} w(k)$$

Eq. (12.3), obtained by augmentation of the state vector, defines a vector Markov sequence (i.e. a vector relation between two successive points of a sequence driven by white noise). A similar augmentation of the state vector can be used for non-Markov processes. For example, let the state equation be

$$\ddot{X}_i = C_1(t)\dot{X}_1 + C_2(t)X_1 + w(t) \qquad (12.18)$$

This is a non-Markov process (scalar or vector), that is, a process which defines a relation between three points (as seen from its finite difference form). To augment the state vector we let $\dot{X}_1 = X_2$, then

$$\underline{\dot{X}}(t) = \underline{\underline{C}}(t)\underline{X}(t) + \underline{w}(t) \qquad (12.19)$$

where

$$\underline{X} = \begin{bmatrix} X_1, X_2 \end{bmatrix}^T \quad ; \quad \underline{\underline{C}}(t) = \begin{bmatrix} 0 & 1 \\ C_2 & C_1 \end{bmatrix} \quad ; \quad \underline{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} w(t)$$

Eq. (12.19) is a vector Markov process, i.e. a vector first order differential relation between random X at two successive points, the system X being driven by white noise.

13. Solutions of the Random Wave Equations

The literature devoted to solution of the random wave equation is enormous. The great profusion of efforts in this research is due to the multitude of distinct disciplines which are concerned with propagation of waves, such as: (a) radio propagation (b) seismology (c) shell and plate dynamics (d) acoustic propagation in the atmosphere and in the ocean (e) astronomical radio telescopes (f) ionosphere studies (g) nucleonics (h) plasma physics (i) optics. We will be concerned in this report only with those solution methods which appear to have relevance to the propagation of sound in the ocean. The number of these methods that can be reviewed here in Part I necessarily limited.
A further discussion of methods is being prepared for Part II of this series.

14. Methods of Solution in Abstract Form

The equations of displacement u_k of an anisotropic medium driven by force P_i with combined deterministic and random inhomogeneities (assumed small) can be written in the form

$$\sum_{k=1}^{3} \left\{ \left(\overset{o}{L}_{ik} - I_{ik}\overset{o}{\varrho}\frac{\partial^2}{\partial t^2} \right) \right.$$

$$+ \epsilon \left(\overset{(1)}{L}_{ik}(\underline{x}) - I_{ik}\varrho_1^o(\underline{x})\frac{\partial^2}{\partial t^2} \right)$$

$$\left. + \delta \left(\overset{(1)}{L}_{ik}(\underline{x},\alpha) - I_{ik}\varrho_1^o(\underline{x},\alpha)\frac{\partial^2}{\partial t^2} \right) \right\} u_k = -P_i, \tag{14.1}$$

$$i = 1, 2, \cdots$$

Here ϵ is a measure of the strength of the deterministic inhomogeneities and δ is a measure of the random inhomogeneities. The symbol $\overset{o}{L}_{ik}$ is a differential form with constant elastic coefficients A_{ik} describing the anisotropic property of the medium. The symbol $\overset{(1)}{L}_{ik}$ is also a differential form with elastic coefficients a_{ik} which are functions of coordinates, describing deterministic inhomogeneities. The symbol $\overset{(1)}{L}_{ik}(\underline{x},\alpha)$ has the same form as $\overset{(1)}{L}_{ik}$ but is a random operator. The density ϱ is defined as the sum of the densities of a homogeneous medium ϱ^o, added deterministic inhomogeneities $\varrho_1^o(\underline{x})$ and added random inhomogeneities $\varrho_1^o(\underline{x},\alpha)$, α being the realization parameter. The solutions u_k of Eq. (14.1) are assumed to have the form

$$u_k = u_k^o + \epsilon u_k' + \delta u_k' \tag{14.2}$$

$$\epsilon u_k' \ll u_k^o$$

$$\delta \ll \epsilon$$

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Substituting (14.2) into (14.1), then collecting terms of the same order of magnitude lead to the following,

order zero:
$$\sum \left(\overset{0}{L}_{ik} - \varrho^0 \, I_{ik} \frac{\partial^2}{\partial t^2} \right) \overset{0}{u}_\kappa = - P_i$$

first order in ϵ :
$$\left(\overset{(1)}{L}_{ik}(\underline{x}) - I_{ik} \, \varrho^0_1 \frac{\partial^2}{\partial t^2} \right) \overset{0}{u}_\kappa =$$
$$- \left(\overset{0}{L}_{ik}(\underline{x}) - I_{ik} \, \varrho^0 \frac{\partial^2}{\partial t^2} \right) \overset{1}{u}_\kappa$$

first order in δ :
$$- \left(\overset{(1)}{L}_{ik}(\underline{x}, \alpha) - I_{ik} \, \varrho^0_1 (\underline{x}, \alpha) \frac{\partial^2}{\partial t^2} \right) \overset{0}{u}_\kappa \qquad (14.3)$$
$$= \left(\overset{0}{L}_{ik} - I_{ik} \, \varrho^0 \frac{\partial^2}{\partial t^2} \right) \overset{1}{u}_\kappa$$

A different set of equations is obtainable from (14.1) by grouping all operators into a deterministic set and a random set, and writing

$$u_\kappa = \overset{0}{u}_\kappa + \delta \, \overset{1}{u}_\kappa \qquad (14.4)$$

The result to different order is:

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order zero: $\displaystyle\sum_{k=1}^{3} \left[\overset{0}{L_{ik}} - I_{ik}\varsigma^0 \frac{\partial^2}{\partial t^2} + \epsilon \left(\overset{(1)}{L_{ik}} - I_{ik}\varsigma_1^0(\underline{x})\frac{\partial^2}{\partial t^2} \right) \right] u_k^0 = -P_i'$

first order in δ : $\displaystyle\sum_{k=1}^{3} \left[\overset{0}{L_{ik}} - \varsigma^0 \frac{\partial^2}{\partial t^2} + \epsilon \left(\overset{(1)}{L_{ik}} - I_{ik}\varsigma_1^0(\underline{x})\frac{\partial^2}{\partial t^2} \right) \right] u_k'$

$$= - \quad \left(\overset{(1)}{L_{ik}}(\underline{x},\alpha) - I_{ik}\varsigma_1^0(\underline{x},\alpha)\frac{\partial^2}{\partial t^2} \right) u_k^0$$

(14.5)

For brevity we set,

$$\overset{0}{\mathcal{L}_{ik}} \equiv \overset{0}{L_{ik}} - I_{ik}\varsigma^0 \frac{\partial^2}{\partial t^2}$$

$$\mathcal{M}_{ik} \equiv \overset{0}{L_{ik}} - I_{ik}\varsigma^0 \frac{\partial^2}{\partial t^2} + \epsilon \left(\overset{(1)}{L_{ik}}(\underline{x}) - I_{ik}\varsigma_1^0(\underline{x})\frac{\partial^2}{\partial t^2} \right)$$

$$\mathcal{L}_{ik}^{(1)}(\underline{x},\alpha) \equiv \overset{(1)}{L_{ik}}(\underline{x},\alpha) - I_{ik}\varsigma_1^0(\underline{x},\alpha)\frac{\partial^2}{\partial t^2}$$

(14.6)

Let $(\overset{0}{\mathcal{L}_{ik}})^{-1}$ be an inverse operator. Then the solutions to Eq. (14.5) are

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$$u_k^0 = -\sum_i \left(\mathcal{M}_{ki}^0\right) P_i \tag{14.7}$$

$$u_k^{(1)} = -\sum_{i,\ell} \left(\mathcal{M}_{ki}^0\right)^{-1} \left[L_{i\ell}^{(1)}(\underline{x},\alpha) - I_{i\ell}(\underline{x},\alpha)\frac{\partial^2}{\partial t^2} \right] u_\ell^0 \tag{14.8}$$

The inverse operator is the deterministic Green's dyadic,

$$\sum_k \mathcal{M}_{ik} G_{k\ell} = I_{i\ell}\, \delta\left(\underline{x}-\underline{x}', t-\tau\right) \tag{14.9}$$

$$\sum_k \mathcal{L}_{ik}^0 G_{ki}^0 = I_{i\ell}\, \delta\left(\underline{x}-\underline{x}', t-\tau\right) \tag{14.10}$$

We now can construct two alternative solutions to Eq. (14.1). First we use operator $\mathcal{L}_{ik}^0$ and its inverse $G_{ki}^{(0)}$. Then the solution u_k is symbolically given by,

$$u_k = -\int_\tau\int_V G_{ki}^0 P_i\, dV d\tau - \epsilon \int_\tau\int_V G_{ki}^0 \left[L_{i\ell}^{(1)}(\underline{x}) - I_{i\ell}\varrho_i^0(\underline{x})\frac{\partial^2}{\partial t^2} \right] u_\ell\, dV d\tau$$

$$- \delta \int_\tau\int_V G_{ki}^0 \left[L_{i\ell}^{(1)}(\underline{x},\alpha) - I_{i\ell}\varrho_i^0(\underline{x},\alpha,t)\frac{\partial^2}{\partial t^2} \right] u_\ell\, dV d\tau \tag{14.11}$$

Secondly we use operator $\mathcal{M}_{ik}$ and its inverse $G_{k\ell}$. Then the solution u_k is

$$u_K = -\int_\tau\int_V G_{ki} P_i\, dV_i d\tau - \delta\int_\tau\int_V G_{ki}\left[L_{i\ell}^{(1)}(\underline{x},\alpha) - I_{i\ell}\varrho_i^0(\underline{x},\alpha)\frac{\partial^2}{\partial t^2} \right]$$

$$\times u_k\, dV_i d\tau \tag{14.12}$$

Conclusion: The symbolic solutions of Eq. 14.1 noted in this section are useful only when the inverse operators G_{ki} etc. can be specifically identified. Several specific cases will be identified and discussed in later Sections of this report.

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15. Perturbation Procedures For the Random Wave Equation in Abstract Notation

We will illustrate the solution of the propagation of sound in a random medium by perturbation procedures in the following schematic listing of formulas and assumptions.

Let $u(\underline{x}, t)$ represent a scalar field variable and $\mathcal{L}$ represent an operator. Then the equation of propagation of sound in the absence of true sources is given by

$$\mathcal{L}(\underline{x}, t) \left\{ u(\underline{x}, t) \right\} = 0 \tag{15.1}$$

Assumption No. 1: $\mathcal{L}$ is linear.

Assumption No. 2: $\mathcal{L}$ is random.

As a result of Assumption No. 2 the field u is a random process.

Assumption No. 3: $\quad u(\underline{x}, t) = V(\underline{x}) e^{-j\omega t}$

Assumption No. 4: $\quad v(\underline{x}, t) = V_c(\underline{x}) + V_i(\underline{x})$

According to Assumption No. 4 the random field can be represented as the sum of a coherent wave $V_c = \langle V \rangle$ and an incoherent wave $V_i = V - \langle V_c \rangle$ where by definition, $\langle V_i \rangle \equiv 0$. The mutual coherence function, M, is defined as

$$M(\underline{x}_1, \underline{x}_2) = \langle u(\underline{x}_1, t) u^*(\underline{x}_2, t) \rangle$$

$$= V_c(\underline{x}_1) V_c(\underline{x}_2) + \langle V_i(\underline{x}_1) V_i^*(\underline{x}_2) \rangle \tag{15.2}$$

Assumption No. 5: there is a plane wave $\exp(-j\omega t + j \underline{k} \cdot \underline{r}), \; z = 0$

Assumption No. 6: (a) Medium is statistically homogeneous in x, y.

(b) Wave propagates in direction z.

Using the notation $(x, y, z) = (\underline{q}, z)$ it is seen that Eq. (2) plus Assumptions No. 5 and No. 6 lead to the mutual coherence function in the plane $z = L$,

$$M(q_1, q_2, L) = \exp(-2k_i L) + \langle V_i(q_1, L) V_i^*(q_2, L) \rangle \tag{15.3}$$

Assumption No. 7: All operators and field quantities can be expanded in powers of a small quantity ϵ.

According to Assumption No. 7 Eq. (15.1) can be expanded in powers of ϵ. The result is

$$\mathcal{L}_1 \{ V_i \} = \epsilon \mathcal{L}_2 \{ V_i \} + \epsilon \mathcal{L}_3 \{ V_c \} + \mathcal{O}(\epsilon^2)$$

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$$\tag{15.4}$$

Since v_i' is $\mathcal{I}(\epsilon)$ and α_2 is $\mathcal{O}(\epsilon)$ it is seen that $\epsilon \mathcal{L}_2 \{ v_i \}$ is negligible. Thus

$$\mathcal{L}_1 \{ v_i \} = \epsilon \mathcal{L}_3 \{ v_c \} \tag{15.5}$$

Taking into account the boundary conditions the solution of Eq. (15.5) is

$$v_i = \mathcal{L}_1^{-1} \epsilon \mathcal{L}_3 \{ v_c \} \tag{15.6}$$

<u>Assumption No. 8</u>: $\mathcal{L}^{-1}$ is nonrandom; $\mathcal{L}_3$ is random.

According to Assumption No. 8, the mutual coherence function using Eq. (15.6) is

$$\langle v_i (\underline{x}_1) v_i^* (\underline{x}_2) \rangle = \epsilon^2 \mathcal{L}_1^{-1} (\underline{x}_1, \underline{x}_1') \mathcal{L}_1^{-1} (\underline{x}_2, \underline{x}_2')$$

$$\times \langle \mathcal{L}_3 (\underline{x}_1') \mathcal{L}_3 (\underline{x}_2') \rangle v_c (\underline{x}_1') v_c (\underline{x}_2') \tag{15.7}$$

Eq. (15.7) is a double integral over 3-space, i.e. six coordinates to be integrated.

<u>Assumption No. 9</u>: $\mathcal{L}_3$ operates on a statistically homogeneous process and is a scalar.

According to Assumption No. 9,

$$\langle \mathcal{L}_3 (\underline{x}_1) \mathcal{L}_3 (\underline{x}_2) \rangle = C (\underline{x}_1 - \underline{x}_2) \tag{15.8}$$

<u>Assumption No. 10</u>: $\mathcal{L}_1^{-1}$ can be approximated by choosing various acoustic sizes for the correlation-length ℓ of the medium inhomogeneities.

Customarily $k_0 \ell$ is chosen $\gg 1$, i.e. small angle scattering.

<u>Assumption No. 11</u>: Choose small angle scattering.

According to Assumption No. 11 the volume integration is limited to cones with vertices at $\underline{q}_1, \underline{q}_2$ with apertures $(k_0 \ell)^{-1}$. The solution of Eq. (15.7) based on Assumptions Nos. 9, 10, 11 is facilitated by introduction of relative and center-of-mass coordinates, i.e.

relative: $\quad \underline{x} = \underline{x}' - \underline{x}''$

center-of mass: $\quad \underline{X} = \frac{1}{2} (\underline{x}' + \underline{x}'')$

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Thus the six-fold guadrature is over $d^3\underline{x}\, d^3\underline{X} = dx\,dy\,dz\,dX_1\,dY_1\,dZ$. Integration over $dXdY$ is easily effected. Integration over $dxdy$ is by method of stationary phase. I tegration over dZ is easily effected. Thus the autocorrelation of V_i is reduced to a single quadrature over dz. Since Eq. (15.7) is solved one can directly obtain M from Eq. (15.2). Perturbation with Taylor expansions are discussed further in Sect. 34, 35.

Survey of Solutions of the Eq. $\nabla^2 u + k_o^2\, n(\underline{r}, t)\, u = 0$

16. Born, Parabolic Approximations

It is first assumed that the medium is homogeneous and stationary,

$$\langle n \rangle = const. \tag{16.1a}$$

$$n(\underline{r}) = \langle n \rangle \left[1 + \tilde{n}(\underline{r}) \right] \tag{16.1b}$$

We begin by restricting $\tilde{n}(\underline{r})$ to be "small enough" and seek a solution by simple perturbation theory,

$$u(\underline{r}, \varepsilon) = u_o(\underline{r}) + \varepsilon\, u_1(\underline{r}) + \varepsilon^2 u_2(\underline{r}) + \cdots \tag{16.2}$$

in which u_n is independent of ε and u_o is the solution when $\varepsilon = 0$. Writing the equation to be solved in the form

$$\nabla^2 u + k^2 u = -k^2 \tilde{n}\, u, \qquad k^2 = k_o^2 \langle n \rangle \tag{16.3}$$

then transforming to an integral equation, then substituting (16.2) leads to the perturbation series (Born approximation),

$$u(\underline{r}) = u_o(\underline{r}) - k^2 \int g(\underline{r}, \underline{r}_1)\, \tilde{n}(\underline{r}_1)\, u_o(\underline{r}_1)\, d^3\underline{r}_1$$

$$+ k^4 \iint g(\underline{r}, \underline{r}_1)\, \tilde{n}(r_1)\, g(\underline{r}_1, \underline{r}_2)\, \tilde{n}(\underline{r}_2)\, u_o(\underline{r}_2)\, d^3\underline{r}_1\, d^3\underline{r}_2 - \cdots \tag{16.4}$$

to be uniformly valid this expansion requires that each term be a small correction to the previous term for all values of $\underline{r}$. In the scattering process this means that the energy scattered over path $x = L$ must be much smaller than the incident energy at $x = 0$. The intensity of energy $I \sim \langle |u|^2 \rangle$ can be calculated if the autovariance (or correlation

function) R of $\tilde{m}(r)$ is known. Choosing a Gaussian distribution with correlation distance ℓ and restricting $u(r)$ of Eq. (16.4) to two terms on the r.h.s. it can be shown that the "smallness condition" limits both the path length L over which the expansion is valid, and the maximum variance $\langle \tilde{m}^2 \rangle$, by the relation,

$$\langle \tilde{m}^2 \rangle \, kL \ll \frac{1}{k\ell \left(1 - e^{-2k^2/k^2}\right)} \tag{16.5}$$

Validity of the perturbation expansion thus requires every path length be smaller than the extinction length d,

$$d = \frac{1}{k \langle \tilde{m}^2 \rangle} \frac{1}{k\ell \left(1 - e^{-2k^2/k^2}\right)} \tag{16.6}$$

When $k\ell$ is small the scattering is omnidirectional and little energy, once scattered, is scattered a second time in the direction of propagation. At every point along the path the incident field is the undisturbed field (viz. u_o).

The "blob" size of inhomogeneity is constant. Eqs. (16.5) and (16.6) therefore apply only to homogeneous and isotropic media. If there is a range of size of inhomogeneities along the path, the medium is no longer homogeneous. In this case it is useful to assume that incremental changes in n from point to point are stationary. The medium is then locally homogeneous over some range of sizes, say from the inner scale ℓ_o to the outer scale L_o. The correlation function, which is the averaged product $\langle \tilde{m}(r_1) \tilde{n}(r_2) \rangle$ and is therefore a maximum when $r_1 = r_2$, emphasizes the large scale inhomogeneities L_o. For a locally homogeneous and isotropic medium it is desirable to eliminate this trend by use of structure functions which are the squares of the averaged differences $\langle \left[n(r_1) - n(r_2) \right]^2 \rangle$. The structure function depends only on $|r_1 - r_2|$ whereas the correlation function may depend on r_1, r_2 separately. For a medium which is everywhere homogeneous and isotropic the measurement of the correlation function of $\tilde{n}(r_1)$ and the calculation of its corresponding power-law

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spatial spectral density are adequate. For a medium which is <u>locally</u> homogeneous and isotropic it is more useful to measure the structure function and calculate the power-law spatial spectral density from it.

When the parameter kl increases the incident wave is scattered more nearly forward and is therefore subject to repeated scattering. Inasmuch as all multiple scattering problems must be solved by approximation the case $(k\ell)^{-1} \ll 1$ may be treated by first setting $u(\underline{x}) = U(\underline{x}) \exp(-ikx)$ where x is the direction of propagation, and then neglecting $\partial^2 U(x)/\partial x^2$ compared to $2k|\partial U(x)/\partial x|$. The result is the parabolic (or diffusion) approximation.

$$-2ik \frac{\partial U(\underline{x})}{\partial x} + \nabla_\perp^2 U(\underline{x}) = -k^2 \tilde{m}(\underline{x}) U(\underline{x})$$

$$(16.7)$$

In this formula the r.h.s. appears as a source of "negative" energy in a diffusion process described by the left hand side. For a point source $-4\pi \delta(\underline{\varrho} - \underline{\varrho}_0)$ in the plane x = const., $\varrho = y\hat{\jmath} + z\hat{k}$, the Green's function for a diffusion process is

$$G_D = \alpha \exp\left\{-ik\left[(y-y_0)^2 + (z-z_0)^2\right]/2x\right\} H(x)$$

$$(16.8)$$

in which H(x) is the Heaviside step function.

Thus, for an incident wave $U(\underline{x})$, the solution of (16.7) may be written in the form of an integral equation,

$$U(\underline{x}) = U_0(\underline{x}) - \alpha k^2 \int_{-\infty}^{\infty} \tilde{n}(\underline{\varrho}_0) U(\underline{\varrho}_0) \exp\left\{\frac{-ik|\underline{\varrho}-\underline{\varrho}_0|^2}{2x}\right\} d^2\underline{\varrho}_0$$

$$(16.9)$$

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Let the integration be over the size of the inhomogeneity, i.e. over $|\varsigma - \varsigma_0|_{MAX} \approx \ell$. Then the ratio

$$\frac{k}{2x}|\varsigma - \varsigma_0|^2 = \mathcal{O}\left(\frac{\ell^2}{\lambda x}\right) \tag{16.10}$$

$$= \mathcal{O}\left(\frac{\text{area of inhomogeneity}}{\text{area of first Fresnel zone}}\right)$$

If the path length x were indefinitely large then the scattering would be independent of direction (i.e. of ς), contrary to the assumptions of forward scattering. The assumption of high directionality in the forward direction requires that the magnitude of the exponent not fall below a certain value. Using $(k\lambda)^{-1}$ as an appropriate scale of smallness, one specifies this value by an inequality.

$$\frac{\ell^2}{\lambda x} \gg \frac{1}{(\varrho \ell)^2} \; ; \qquad \text{or} \qquad \frac{\ell^2}{\lambda x} \gg \frac{\lambda^2}{\ell^2} \tag{16.11}$$

Thus the path x over which the application of the parabolic equation is valid is limited by Eq. (16.11). This equation is seen to allow the following choices:

Assumption: $k\ell \gg 1$

Choices: (a) $\dfrac{\ell^2}{\lambda x} \gg 1$, or 'geometric optics' approximation

(b) $\dfrac{\ell^2}{\lambda x} \sim 1$, or Fresnel approximation

(c) $\dfrac{1}{(k\ell)^2} \ll \dfrac{\ell^2}{\lambda x} \ll 1$, or Fraunhofer approximation

Since ℓ/λ can be much larger than unity it is seen that ℓ/x can be much less than unity. that is. the parabolic approximation permits many inhomogeneities to be in the direct path of propagation. It is to be noted however that Eq. (16.9) is an integral

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equation in U , and hence is not directly solvable by quadrature. The appearance of the solution for the field in the form of an integral equation is a feature of the parabolic equation method.

17. The method of Smooth Perturbations (MSP)

In the reduced wave equation for a medium $n(\underline{x}) = n_0(\underline{x}) + \delta n_1(\underline{x})$,

$$(\nabla^2 + k_0^2 n_0^2) U(\underline{x}) = -(k_0^2 2 n_0 \delta n_1 + k_0^2 \delta n_1^2) U(\underline{x})$$

(17.1)

the method of smooth perturbations (or the Rytov method) seeks a solution of the form

$$U(\underline{x}) = \exp[\psi_0(\underline{x}) + \psi_1(\underline{x}) + \cdots]$$

(17.2)

in which $\exp(\psi_0(\underline{x}))$ is the solution when $\delta n_1 = 0$. The proposed solution leads to the following nonlinear d.e. in $\psi_1(\underline{x})$,

$$\nabla^2 \psi_1(\underline{x}) + \nabla \psi_1(\underline{x}) \cdot [2\nabla \psi_0(\underline{x}) + \nabla \psi_1(\underline{x})]$$

$$+ 2 k_0^2 \delta n_0(\underline{x}) n_1(x) + k^2 \delta n_1^2(\underline{x}) = 0$$

(17.3)

Assuming that $\psi_1(\underline{x})$ possesses an asymptotic expansion in powers of the small parameter δ :

$$\psi_1(\underline{x}) = \delta \sum_{j=0}^{\infty} \delta^j \psi_{1j}(\underline{x})$$

(17.4)

One can reduce Eq. (17.3) to a recursive system of d.e. in ψ_{1j}' , whose m'th term is

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$$(\nabla^2 + 2 \underline{\nabla} \psi_0 \cdot \underline{\nabla}) \; \psi_{1m} \; (\underline{x}) = \chi (\underline{x}) \qquad (17.5)$$

$$\chi_0 \; (\underline{x}) = -2 k_0^2 n_0 (\underline{x}) n_1 (\underline{x}) \qquad (17.6)$$

$$\chi_1 \; (\underline{x}) = -k_0^2 n_1^2 (\underline{x}) - \nabla \psi_{10} \cdot \nabla \psi_{10} \qquad (17.7)$$

$$\chi_m \; (\underline{x}) = - \sum_{p=0}^{m-1} \nabla \psi_{1p} \; (\underline{x}) \cdot \nabla \psi_{1, m-p-1} (\underline{x}) \qquad (17.8)$$

To solve this recursive set the random field $n_1 \; (x)$ is assumed to be homogeneous and isotropic, representable by a two-dimensional Fourier-Stieltjes spectral integral in the random process dN,

$$n_1 (\underline{x}) = \iint \exp (i \; \underline{K}_T \cdot \underline{x}_T) \, dN \; (\underline{K}_T, \alpha) \qquad (17.9)$$

$$\underline{K}_T = \hat{e}_y \; k_y + \hat{e}_z \; k_z$$

$$\underline{x}_T = \hat{e}_y \; y + \hat{e}_z \; z$$

where

$$< dN \; (\underline{K}_T, x) \, dN^* (\underline{K}_T', x') > = \delta (\underline{K}_T - \underline{K}_T') \, F_m (\underline{K}_T | x - x'|) \, d\underline{K}_T \, dK_T' \qquad (17.10)$$

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The symbol $\bar{r}_n$ is the 2-dimensional spectral density of $n_1(\underline{x})$. The stochastic fields ψ_{ij} and χ also have spectral representations,

$$\psi_{ij} = \iint \exp\left[i\,\underline{\mathcal{K}}_T \cdot \underline{x}_T\right] d\underline{\Psi}_j(\underline{\mathcal{K}}_T, x) \qquad (17.11)$$

$$\chi_j(\underline{x}) = \iint \exp\left[i\,\underline{\mathcal{K}}_T \cdot \underline{x}_T\right] dX_j(\underline{\mathcal{K}}_T, x) \qquad (17.12)$$

From (17.6), (17.9) and (17.12) it is seen that for j = o in (17.4)

$$dX_o = -2\,k_o^2\,dN(\underline{\mathcal{K}}_T, x) \qquad (17.13)$$

If the field $n_1(\underline{x})$ is finite in some interval bounded $0 < x' < x$ and goes to zero outside of this interval, and if the Fresnel approximation to the Green's function is used then the random amplitude $d\underline{\Psi}$ in Eq. (17.11) becomes

$$d\underline{\Psi}_o(\underline{\mathcal{K}}_T, x) \sim i\,k_o \int_0^x dx_1\, \exp\left\{\frac{i\,\mathcal{K}^2[x'-x]}{2\,k_o}\right\} dN(\underline{\mathcal{K}}_T, x') \qquad (17.14)$$

Similarly, under the same assumptions

$$d\underline{\Psi}_1(\underline{\mathcal{K}}_T, x) \sim \frac{ik}{2} \int_0^x dx' \exp\left\{\frac{i\,\mathcal{K}^2[x'-x]}{2\,k_o}\right\} \iint_{-\infty}^{\infty} dN(\underline{\mathcal{K}}_T - \underline{\mathcal{K}}_{T_o}', x')\, dN(\underline{\mathcal{K}}_{T_o}', x')$$

$$+ \frac{i}{2k_o} \int_0^\infty dx' \exp\left\{\frac{i\,\mathcal{K}^2(x'-x)}{2k_o}\right\} \iint_{-\infty}^{\infty} (\underline{x} - \underline{\mathcal{K}}_T') \cdot \underline{\mathcal{K}}_T'\, d\underline{\Psi}_o(\underline{\mathcal{K}}_T - \underline{\mathcal{K}}_{T_o}', x') \qquad (17.15)$$

$$\times d\underline{\Psi}_o(\underline{\mathcal{K}}_T', x').$$

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Eqs. (17.14) and (17.15) allow one to calculate ψ_{10} and ψ_{11} from Eq. (17.11) respectively. The second moment of these two quantities can then be formed and then used to form a criterion of the limit for the validity of the Rytov approximation. Brown [J. Opt. Soc. Am. 56, 1045, (1966)] has adopted the following criterion,

$$\delta^2 \langle \psi_{11}(\underline{x}) \psi_{11}^{x}(\underline{x}) \rangle << \langle \psi_{10}(\underline{x}) \psi_{10}^{*}(\underline{x}) \rangle \quad (17.16)$$

He assumes that $M_1(x)$ is a gaussian random process with a spectral representation of the second moment

$$F_m \left(\underline{K_T}, |X_i - X_j| \right) = \frac{\ell^2}{4\pi} exp \left[-\frac{K_T^2 \ell^2}{4} - \frac{|X_i - X_j|^2}{\ell^2} \right] \quad (17.17)$$

in which ℓ is the correlation distance of fluctuations. Based on these assumptions he arrives at the conclusion that MSP is valid when

$$\pi^{\frac{1}{2}} \delta^2 (k\ell)^3 \gamma \left[\frac{1}{4} T(\gamma) + \frac{1}{256} \frac{P(\gamma)}{\gamma^2} \right] < 1 \quad (17.18)$$

$$\gamma = \frac{x}{(k\ell)^2}$$

$$T = \left[1 - \frac{\gamma}{4} \tan^{-1} 4\gamma \right]^2 + \frac{1}{64\gamma^2} \left[\ln\left(1 + 16\gamma^2 \right) \right]^2$$

(17.19)

$$P = 192\,\gamma^2 - 144\,\gamma \tan^{-1} 2\gamma + 28 \ln\left(4\gamma^2 + 1 \right)$$

$$- \frac{140}{3} + \frac{49/3}{(\gamma^2 + 1/4)} + \frac{(128/3\gamma^4) + 6\gamma^2 - 7/6}{(\gamma^2 + \frac{1}{4})^2}$$

(17.20)

<u>Discussion:</u> Brown has discussed his results in the following way

(1) When $\gamma \gg 1$ it is required that

$$X < \frac{1}{\pi^{\frac{1}{2}} k^2 \ell \delta^2}$$

(17.21)

This is the same requirement for validity of the first order Born approximation. Hence Brown surmises that both the Rytov (MSP) and Born approximations have the same domain of validity for all γ .

(2) In the geometric optics approximation (with solution $u = A e^{iS}$, A is the amplitude and S is the phase of the wave) governed by the equations

$$\underline{\nabla} S \cdot \underline{\nabla} S = k_o^2 n^2(x)$$

(17.22)

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$$\nabla^2 S + 2\nabla \ln A \cdot \nabla S = 0 \tag{17.23}$$

$$S = S_0 + S_1 \tag{17.24}$$

$$\ln A = \ln A_0 + \Lambda \, , \qquad \Lambda = \frac{\ln A}{A_0} \tag{17.25}$$

the mean square value of S_1 and Λ have been derived by Tatarski [Wave Prop in Turb. Media, p. 148, (1961)] to be

$$\langle S_1^2 \rangle = \pi^{\frac{1}{2}} \delta^2 k^2 x \ell \tag{17.26}$$

$$\langle \Lambda^2 \rangle = \frac{8}{3} \pi^{\frac{1}{2}} \frac{\delta^2 x^3}{\ell^3} \tag{17.27}$$

The first term in the geometric optics solution is related to $\langle S_1^2 \rangle$, and the second term to $\langle \Lambda^2 \rangle$. Using the equivalent of (17.16) leads to the result that in the geometric optics case,

$$x < 0.61 \, k\ell^2 \qquad or \qquad \gamma < 0.61 \tag{17.28}$$

The MSP approximation in contrast is valid for all γ subject to (17.18).

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18. The Dyson Equation

The basic differential equation of propagation in a random medium is written symbolically as a stochastic equation in the random field ψ resulting from scattering by the random inhomogeneitis

$$L_o \psi = \xi(\underline{r})\psi + f(\underline{r}) \tag{18.1}$$

The solution of (18.1) by perturbation theory is the Neumann series,

$$\psi = \sum_o^\infty (M_o \xi)^n M_o f, \qquad M_o = L_o^{-1} \tag{18.2}$$

(i.e. the Born solution). The first term is the unperturbed source wave, the second term is the singly scattered wave, etc. Thus the lowest order statistical description $\langle\psi\rangle$ (= the mean value) requires a knowledge of the moments of ξ of all orders. Let $\psi = \langle\psi\rangle + \phi$, $\langle\phi\rangle = 0$, namely the stochastic field is separated into a mean (= coherent) part and a fluctuating (= incoherent) part. Substituting for ψ into (18.1) and then averaging one obtains

$$L_o \langle\psi\rangle = \langle\xi\phi\rangle + f, \qquad \langle\xi\rangle \equiv 0 \tag{18.3}$$

The objective is to write $\langle\xi\phi\rangle$ as a function of $\langle\psi\rangle$. Subtracting (18.3) from (18.1) gives

$$L_o \phi = \xi\langle\psi\rangle + \xi\phi - \langle\xi\phi\rangle \tag{18.4}$$

The solution of (18.4) is

$$\phi = M_o\left[\xi\langle\psi\rangle\right] + M_o\left[\xi\phi - \langle\xi\phi\rangle\right]$$

By iteration,

$$\phi = M_o\left[\xi\langle\psi\rangle\right] + M_o\left[\xi M_o \xi\langle\psi\rangle - \langle\xi M_o \xi\langle\psi\rangle\rangle + \cdots\right. \tag{18.5}$$

Multiplying by ξ and averaging leads to

$$\langle\xi\phi\rangle = \langle\xi M_o\left[\xi\langle\psi\rangle\right]\rangle + \left[\langle\xi M_o \xi M_o \xi M_o \xi\rangle\right.$$

$$\left. - \langle\xi M_o \xi\rangle M_o \langle\xi M_o \xi\rangle + \cdots\right]\langle\psi\rangle \tag{18.6}$$

$$\text{''} \qquad = Q\langle\psi\rangle$$

in which Q is the "medium description operator." Eq. (18.6) states that the mean value of the coupling between the incoherent field and the random index of refraction depends on the mean field through on infinite (non-summable) series of operators. Substitution of (18.6) into (18.3) leads to,

$$(L_0 - Q) \langle \psi \rangle = f \tag{18.7}$$

The symbolic solution of (18.7) is

$$\langle \psi \rangle = M_0 f + M_0 Q \langle \psi \rangle, \qquad M_0 = L_0^{-1} \tag{18.8}$$

Another solution is

$$\langle \psi \rangle = M f, \qquad M = (L_0 - Q)^{-1} \tag{18.9}$$

Combining these two solutions leads to the Dyson equation governing random operators,

$$M = M_0 + M_0 Q M \tag{18.10}$$

Thus the operator M obeys an integral equation with Q as kernel. Since Q is not summable Dyson's equation is symbolic only. It is noted from Eq. (18.6) that all terms in Q containing "free" (i.e. unaveraged) M_0 have been subtracted out. Retaining only the first term in Q leads to

$$M_1 = (L_0 - Q_1)^{-1} = \left[L_0 (1 - M_0 \langle \xi M_0 \xi \rangle) \right]^{-1} \tag{18.11}$$

$$" = \sum_{n=0}^{\infty} (M_0 \langle \xi M_0 \xi \rangle)^n M_0 \tag{18.12}$$

Thus not only is (18.9) an infinite sum, but each term itself of this sum is an infinite sum. By going from the random function ψ in (18.2) to its mean value in (18.9) we have re-grouped terms in (18.2) into an infinite sum of sub-infinities. Use of one term in the latter sum is equivalent to selective summation of a sub-infinite of terms in (18.2).

The significance of single and multiple scattering formulations is illustrated by the method and equations of Tatarski and Gerstenstheyn [12]. Beginning with (18.4), and using $M_0 = L_0^{-1}$ they find

$$\phi = K \langle \psi \rangle + K \phi \tag{18.13}$$

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where K is defined by the single scattering operation

$$Kg \equiv M_0 \left[\xi g - \langle \xi g \rangle \right] \tag{18.14}$$

Thus,

$$\langle \xi \varphi \rangle = \langle \xi K \langle \psi \rangle \rangle + \langle \xi K \varphi \rangle \tag{18.15}$$

Substitution of this result in (18.3) leads to

$$\left\{ (L_0 - \langle \xi K \rangle) \right\} \langle \psi \rangle = f + \langle \xi K \varphi \rangle \tag{18.16}$$

From (18.14) it is seen that when g is non-random, $Kg \equiv M_0 \xi g$. Thus Eq. (18.16) becomes

$$\langle \psi \rangle = M_1 \left(f + \langle \xi K \varphi \rangle \right) \tag{18.17}$$

$$M_1 = \left(L_0 - \langle \xi M_0 \xi \rangle \right)^{-1} \tag{18.18}$$

In contrast to the symbol K which represents a single scattering, the symbol M_1 represents multiple scattering. Thus Eq. (18.17) states that in a one term approximation to Q, the mean field is determined by selected multiple scattering of the orginal source wave while the fluctuating field is determined by single scattering of the mean wave. Eq. (18.8) when $Q = Q_1$ is equivalent to Bourret's equation [13], which, when explicity written, has the form

$$\langle \psi(\underline{r}) \rangle = \int \mathcal{G}(\underline{r}, \underline{r}') f(\underline{r}') d^3 \underline{r}' + \int_{V_1} G(\underline{r}, \underline{r}_1) \langle \xi(\underline{r}_1) \int_{V_2} G(r_1, r_2)$$

$$\times \; \xi(\underline{r}_2) \rangle \langle \psi(\underline{r}_2) \rangle \, d^3 \underline{r}_1 \, d^3 \underline{r}_2 \tag{18.19}$$

Since $\langle \psi \rangle$ depends only on the two-point statistic of ξ, the approximation is called bilocal. Iteration of (18.8) for $Q_1 = \langle \xi M_0 \xi \rangle$ adds terms with kernels

$$M_0 Q_1, M_0 Q_1, \ldots M_0 f$$

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Thus the calculation of the mean value of the wave field depends on the repeated integration over the two point stastic as contained in Q_1 .

19. The Bethe-Salpeter Equation [14]

From perturbation theory the total field in a random medium is given by

$$\Psi = \sum_{n=0}^{\infty} (M_0 \xi)^n M_0 \Delta \tag{19.1}$$

The fluctuating part of the field is therefore

$$\delta \Psi = \sum_{n=0}^{\infty} \left[(M_0 \xi)^n M_0 - \langle (M_0 \xi)^n \rangle M_0 \right] \Delta \tag{19.2}$$

The averaged product or correlation function of the fluctuating field is

$$\langle \delta \Psi(\underline{x}_1) \delta \Psi^*(\underline{x}_2) \rangle = \langle \sum_{n=0}^{\infty} \left[(M_0 \xi)^n M_0 - \langle (M_0 \xi)^n \rangle M_0 \right] \Delta$$

$$\times \sum_{m=0}^{\infty} \left[(M_0 \xi)^{*m} M_0^* - \langle (M_0 \xi)^{*m} \rangle M_0^* \right] \Delta(\underline{x}_1) \Delta(\underline{x}_2) \tag{19.3}$$

Here, M_0 operates on $\underline{x}_1$ coordinates, and M_0^* on the $\underline{x}_2$ coordinates. Explicit terms of this series in the low orders appear as

$$\langle \delta \Psi(\underline{x}_1) \delta \Psi^*(\underline{x}_2) \rangle = \langle (M_0 \xi M_0) \left\{ (M_0 \xi M_0)^* + (M_0 \xi M_0 \xi M_0)^* \right.$$

$$+ (M_0 \xi)^{*3} M_0^* - \langle (M_0 \xi)^{*2} M_0^* \rangle - \langle (M_0 \xi)^{*3} M_0 \cdots \} \rangle$$

$$+ \langle M_0 \xi M_0 \xi M_0 \left\{ (M_0 \xi M_0)^* + (M_0 \xi)^{*2} M_0^* + (M_0 \xi)^{*3} M_0^* \cdots \} \rangle$$

$$+ \cdots \rangle \Delta(\underline{x}_1) \Delta(\underline{x}_2) \tag{19.4}$$

If the sources Δ are point sources then all operators M_0 are converted to Green's functions, i.e.

$$- M_0 \, 4\pi \, \delta(\underline{x} - \underline{x}') = G(\underline{x}, \underline{x}') \tag{19.5}$$

Using point sources, the sequence of terms in the correlation function series may be visualized by double (or parallel line) diagrams drawn according to the following symbolism: a light upper line represents the free-field propagator $G_0(x_i, x_j)$, which is labelled

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X_i at its left end and X_j' at its right end. An upper dot represents $\xi(x_j)$. A light lower line represents $G_0^*(x_\ell, x_p)$ which is labelled X_ℓ at its left end and X_p at its right. A lower dot represents $\xi^*(x_p)$. A dotted line connecting any two dots (in a line, or between lines) represents the process of statistical averaging. To use this symbolism, it is convenient to define the correlation function density, $W(x_1, x_2; r_0, r_0')$ by the formula

$$\langle \delta \Psi(\underline{x}_1)\, \delta \Psi^*(\underline{x}_2) \rangle_{\underline{r}_0} = \iint W(\underline{x}_1, \underline{x}_2; \underline{\varrho}_0, \underline{\varrho}_0')\, \delta(\underline{\varrho}_0 - \underline{\varrho}_0')\, \delta(\underline{\varsigma}_0' - \underline{r}_0')\, d^3\underline{\varrho}_0\, d^3\underline{\varsigma}_0'$$

(19.6)

where $\underline{r}_0$ is the origin of coordinates. The purpose of introducing W is to have each term in the parallel-line series expansion of the correlation function possess four terminals, $\underline{x}_1,\ \underline{\varrho}_0$ in the upper line, and $\underline{x}_2,\ \underline{\varrho}_0'$ in the lower.

By multiplying out $\delta \Psi(x_1)$ and $\delta \Psi^*(x_2)$, then averaging, it is seen that the density can be represented as the following (infinite) sequence,


Double diagrams, which can be broken into two distinct parts by cutting two solid lines (G_0, G_0^*) and no dotted lines, are called weakly coupled (or unconnected) diagrams. The remaining diagrams not so divisible, are called strongly coupled (or connected). The above sequence for W is seen to contain infinite subsequences of both strongly and weakly coupled diagrams. In analogy with the construction of the Dyson equation, it is an objective of further analysis to construct an equation for W which is closed and contains only strongly coupled diagrams, yet which, upon interation, is capable of generating all of the infinite series (both connected and unconnected), by iteration. This is done in the following way. First, an infinite subsequence of strongly coupled diagrams ($= \mathcal{U}$) is isolated from W. These are visualized as the sequence,

Here, the 4-terminal symbol is defined by

$$\begin{array}{cc} r_1 & r_3 \\ \boxed{o} & \\ h_2 & h_4 \end{array} = \; \overset{\circ}{\underset{\circ}{|}} \; + \; \overset{\circ}{\underset{\circ\circ}{|}} \; + \; \overset{\circ\,\,\circ}{\underset{\circ}{\sqcap}} \; + \ldots$$

and contains only strongly connected (i.e. non factorable) diagrams. The appearance of a single upper (or lower) terminal represents a coalescence of terminals (i.e. a delta function). Also, $m+2$ terminals in upper (or lower) line indicates m auxiliary integrations over dummy variables. Next, in lieu of light lines, which represent G, one substitutes dark lines representing the average Green's function $\langle G \rangle$. Then a sequence of diagrams is obtained in which the p'th term has the form

$$\boxed{o}_1 \; \boxed{o}_2 \; \boxed{o}_3 \; \cdots \; \boxed{o}_p$$
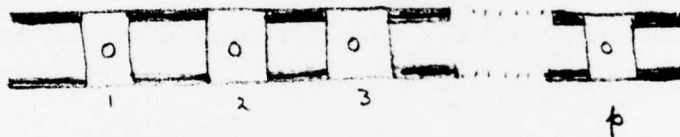
Since the element $\boxed{o}$ is repeated p times in the above diagram, it is seen that this diagram includes all that class of diagrams of W which contain p coupled elements. The sequence $p = 1$ contains an infinite series of strongly coupled diagrams. The sequence $p > 1$ contains infinite series of weakly coupled diagrams. The summation of all p diagrams, $p = 1, 2, \ldots \infty$, thus sums all diagrams and makes up the correlation function density ($= W$). Such a summation can be generated by iteration of the following analogue of the Dyson equation, which contains only strongly connnected diagrams, since each of the two terms contain only $p = 1$,

$$W = \boxed{\times} = \begin{array}{c} x_1 \; r_1 \; r_3 \; r_0 \\ \boxed{o} \\ x_2 \; r_2 \; r_4 \; h_0' \end{array} + \begin{array}{c} x_1 \; r_1 \; r_3 \quad\quad r_0 \\ \boxed{o} \;\; \boxed{\times} \\ x_2 \; h_2 \; r_4 \quad\quad h_0' \end{array}$$

In analytic terms, this diagram (called the analogue of the Beth-Salpeter equation) is written as

$$W(\underline{x}_1, \underline{x}_2 ; \underline{h}_0, \underline{h}_0') = \iiiint \langle G(\underline{x}_1, \underline{h}_1) \rangle \langle G^*(\underline{x}_2, \underline{h}_2) \rangle P(\underline{h}_1, \underline{h}_2 ; \underline{h}_3, \underline{h}_4)$$

$$\times \langle G(\underline{h}_3, \underline{h}_0) \rangle \langle G^*(\underline{h}_4, \underline{h}_0') \rangle \, d^3\underline{h}_1 \, d^3\underline{h}_2 \, d^3\underline{h}_3 \, d^3\underline{h}_4 \quad (19.7)$$

$$+ \iiint \langle G(\underline{x}_1, \underline{k}_1) \rangle \langle G^*(\underline{x}_2, \underline{r}_2) \rangle P(\underline{k}_1, \underline{k}_2; \underline{k}_3, \underline{k}_4) W(\underline{h}_3, \underline{h}_4; \underline{h}_0, \underline{h}_0')$$
$$\times d^3\underline{r}_1 \, d^3\underline{k}_2 \, d^3\underline{r}_3 \, d^3\underline{r}_4 \tag{19.8}$$

in which the intensity operator P is defined analytically as that function satisfying the equation

$$u(\underline{x}_1, x_2; \underline{r}_0, \underline{r}_0') = \iiint G_0(\underline{x}, \underline{x}) G_0^*(x_2, \underline{k}_2) P(\underline{r}_1, \underline{r}_2; \underline{r}_3, \underline{r}_4) G_0(\underline{r}_3, \underline{r}_0) G_0^*(\underline{r}_4, \underline{\Omega}')$$
$$\times d^3\underline{r}_1 \, d^3\underline{r}_2 \, d^3\underline{r}_3 \, d^3\underline{r}_4 . \tag{19.9}$$

The symbol P thus is the <u>undressed diagram</u> (= [◦]) made up of a infinite subsequence of strongly coupled diagrams. In explicit terms

$$P(\underline{k}_1, h_2; \underline{k}_3, \underline{k}_4) = \epsilon^2 |k|^4 \delta(\underline{k}_1 - \underline{r}_3) \delta(\underline{r}_2 - \underline{r}_4) \Gamma(\underline{r}_1, \underline{r}_2)$$
$$+ \epsilon^4 k^2 k^{*6} \delta(r_1 - \underline{r}_3) \Gamma(\underline{r}_2, \underline{r}_4) \int G_0^*(\underline{k}_2, \underline{g}) G_0^*(\underline{k}_4, \underline{y}) \Gamma(\underline{k}_1, \underline{y}) d^3\underline{y} + \cdots$$

in which

$$\Gamma(\underline{r}_1, \underline{k}_2) = \langle \mu(\underline{k}_1) \mu^*(\underline{k}_2) \rangle, \text{ etc.} \tag{19.10}$$

Here it is to be clarified that W requires a quadruple integration over the outer terminals of P while P requires $(m + m)$ "<u>interior</u>" integrations in case there are $m + 2$ terminals in the upper line and $n + 2$ terminals in the lower line. While the analogue of the Bethe-Salpeter equation formally gives the correlation function density expressed as an integral equation, it cannot be solved except by approximation, because P is known only in the form of an infinite series. A first order approximation (= W_1) is obtained when the first term of the series of P is used. In diagramatic form, this approximation (called the "ladder" approximation) is sketched as follows:

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Analytically,

$$W_1 (\underline{x}_1, \underline{x}_2 ; \underline{t}_0, \underline{t}_0') = \epsilon^2 |k|^4 \iint < G (\underline{x}_1, \underline{\xi}_1) > < G^* (\underline{x}_2, \underline{\xi}_2) > \Gamma (\underline{y}_1 - \underline{y}_2)$$

$$< G (\underline{y}_1, \underline{t}_0) > < G^* (\underline{y}_2, \underline{t}_0') > d^3 y_1 d^3 \underline{y}_2 + \epsilon^2 |k|^4 \iint < G (\underline{x}_1, \underline{y}_1) \cdot$$
$$< G^* (\underline{x}_2, \underline{\xi}_2) > \Gamma (\underline{y}_1 - \underline{y}_2) W (\underline{y}_1, \underline{\xi}_2 ; \underline{t}_0, \underline{t}_0') d^3 \underline{y}_1 d^3 \underline{y}_2 . \qquad (19.11)$$

The correlation function density thus relates the second statistical moment of the wave field (= l.h.s.) to the second moment of the random function (= $\Gamma (\underline{y}_1 - \underline{y}_2)$)describing the turbulence (r.h.s.).

20. Wave Numbers of the Medium and Wave Numbers of the Pressure Field (Chernov [9])

By use of the two-dimensional stochastic Fourier-Stieltjes integral one can expand the inhomogeneities of the medium into a spatial wave number series,

$$\epsilon (\underline{x}) = \iint_{-\infty}^{\infty} \exp \left[i (K_2^{(1)} y + K_3^{(1)} z) \right] dN (x, K_2^{(1)}, K_3^{(1)}) \qquad (20.1)$$

$$\underline{x} = (x, y, z)$$

in which the wave number at point 1 is

$$\underline{K}^{(1)} = (K_1^{(1)}, K_2^{(1)}, K_3^{(1)})$$

The random amplitudes $d\nu$ at $x^{(1)}, K_2^{(1)}, K_3^{(1)}$ in the case where the random process ϵ is homogeneous are uncorrelated to any other wave numbers.(say, x, K_2, K_3)This means,

$$< dN (x^{(1)}, K_2^{(1)}, K_3^{(1)}) dN^* (x, K_2, K_3) > = \overline{\epsilon^2} \, \delta (K_2 - K_2^{(1)}) \delta (K_3 - K_3^{(1)})$$

$$F_m (|\underline{x} - \underline{x}^{(1)}|, K_2, K_3) \, dK_2 \, dK_3 \, dK_2^{(1)} \, dK_3^{(1)} \qquad (20.2)$$

Similarly the stochastic wave field $\psi(\underline{X}, t)$ can be expanded to yield,

$$\psi(\underline{X}, t) = \int\int_{-\infty}^{\infty} exp\left[i(\kappa_2 Y + \kappa_3'' Z)\right] d\bar{\Phi}(X, \kappa_2'', \kappa_3''; t)$$

$$\underline{X} = (X, Y, Z) \quad ; \quad k = \omega/c \tag{20.3}$$

The object of analysis is to relate the random properties of the wave field at $\underline{X}$ to the random properties of the medium at $\underline{x}$. As an illustration we use the case of single scattering in the Born approximation. We consider a volume $V_o(\underline{x})$ of inhomogeneities $Q_o(\underline{x})$ and a scalar wave field $p(\underline{X}, t)$. The pressure scattered to $\underline{X}$ by the inhomogeneities of the medium is,

$$p_s(\underline{X}, t) = \frac{-1}{4\pi} \int_{V_o} Q_o(\underline{x}) \frac{e^{\pm ik|\underline{x}-\underline{X}|}}{|\underline{x}-\underline{X}|} d^3\underline{x} \tag{20.4}$$

In the first order Born approximation

$$Q_o(\underline{x}) = \epsilon(\underline{x}) k^2 p_o(\underline{x}, k) \tag{20.5}$$

in which $p_o(\underline{x}, k)$ is the undisturbed incident field. For a point source at the origin with amplitude a_o,

$$p_o(\underline{x}, k) = a_o \, exp\left[\frac{ik|\underline{x}|}{|\underline{x}|}\right] \tag{20.6}$$

Thus the first order scattered pressure is

$$p_s^{(i)}(\underline{X}, k) = \frac{-a_o k^2}{4\pi} \int_{V_o} \epsilon(\underline{x}) \, exp\left[\pm ik|\underline{x}-\underline{X}|\right] \frac{e^{ik|\underline{x}|}}{|\underline{x}-\underline{X}|} \frac{e^{ik|\underline{x}|}}{|\underline{x}|} d^3\underline{x} \tag{20.7}$$

Now

$$|\underline{x}-\underline{X}| = (x^2 + X^2 - 2\underline{x}\cdot\underline{X})^{1/2}$$

If $|\underline{x}| << |\underline{X}|$, then

$$|\underline{x}-\underline{X}| \approx X\left[1 + \frac{x^2}{2X^2} - \frac{\underline{x}\cdot\underline{X}}{X^2}\right] + \mathcal{O}\left(\frac{\underline{x}\cdot\underline{X}}{X^2}\right) \tag{20.8}$$

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The condition that the quadratic term is negligible (i.e., the Fraunhofer condition) is to choose X such that

$$\frac{k x^2}{2X} \ll \frac{\pi}{2} \qquad , \; i.e. \qquad X \gg \frac{kx^2}{\pi} \tag{20.9}$$

If X_{MIN} is at least one correlation length ℓ then the minimum Fraunhofer distance is

$$X_{MIN} \gg \frac{k\ell^2}{\pi} \tag{20.10}$$

If the point of observation X is near the axis of propagation (X-axis) then the Fresnel approximation may be used,

$$|\underline{x}| \simeq x + \frac{y^2 + z^2}{2x}$$

$$|\underline{x} - \underline{X}| \approx X - x + \frac{(Y-y)^2 + (Z-z)^2}{2(X-x)} \tag{20.11}$$

$$\left(|\underline{x}| |\underline{x} - \underline{X}|\right)^{-1} \approx \left(x(X-x)\right)^{-1}$$

Thus the first order scattered pressure in the Fresnel approximation is,

$$p^{(1)}(\underline{X}, k) = -\frac{a_0 k^2 e^{ikX}}{4\pi} \int_{x_1}^{X} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \; \frac{\epsilon(x)}{x(X-x)}$$

$$\cdot \exp ik \left[\frac{X(y^2+z^2) + x(Y^2+Z^2) - 2x(Yy + Zz)}{2x(X-x)} \right] \tag{20.12}$$

The integration over X is limited by neglect of backscatter to the range $X_1 \leqslant x \leqslant X$ in which X_1 is the minimum X containing inhomogeneities. The relation between dN and $d\Phi$ is found by substituting (20.3) on the $\ell.h.s.$ of (20.12) and (20.1) on the $r.h.s.$, multiplying both sides by $\exp\left\{i\left[K_2 Y + K_3 Z\right]\right\} dK_2 dK_3$ and integrating over Y, Z. The result is

$$d\Phi(X, K_2, K_3; k) = dK_2 dK_3 \, e^{ikX} \left(-\frac{a_0 k^2}{4\pi}\right) \int_{x_1}^{X} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz \; \frac{dN(x, K_2', K_3')}{x(X-x)}$$

$$\times \exp ik \left[\frac{X(y^2+z^2) + x(Y^2+Z^2) - 2x(Yy + Zz)}{2x(X-x)} \right]$$

$$\cdot \exp i \left[K_2' y - K_2 Y + K_3' z - K_3 Z \right] dY dZ \tag{20.13}$$

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The integration over $\underset{o}{\nu}$ can be performed using formulas 3.694 (1)(2) of Gradsteyn and Ryzhik [16]. The integration over Y, Z is simplified by choosing $x/X \ll 1$ and using the formula

$$\int_{-\infty}^{\infty} \exp(i\lambda z)\,dz = 2\pi\,\delta(\lambda)$$

The integration over κ_2', κ_3' is trivial. The final result is the random amplitude $d\bar{\Phi}$ expressed as an integral over the propagation coordinate $(= x)$,

$$d\bar{\Phi}(\kappa_2, \kappa_3, \lambda; \varepsilon) = -i\,a_o\,k\,2\pi^2\,(e^{ikX}/X)\int_{x_i}^{X} dx$$

$$\exp\left[-i\,\frac{X(X-x)(\kappa_2^2+\kappa_3^2)}{2kx}\right]\,dN\left(\kappa_2\,\frac{X}{x}\,,\,\kappa_3\,\frac{X}{x}\,,\,x\right) \quad (20.14)$$

in which we have used the formula

$$d\kappa_2'\,d\kappa_3'\,\frac{X^2}{x^2}\int_{-\infty}^{\infty}\delta\left(\kappa_2\,\frac{X}{x}-\kappa_2'\right)\delta\left(\kappa_3\,\frac{X}{x}-\kappa_3'\right)\,dN(\kappa_2',\kappa_3',x)$$

$$\qquad\qquad\qquad\qquad\qquad\qquad (20.15)$$

$$= dN\left(\kappa_2\,\frac{X}{x}\,,\,\kappa_3\,\frac{X}{x}\,,\,x\right)$$

The significance of Eq. (20.14) is this. This spatial inhomogeneities of the wave characterized by the wave number κ_2 (or dimension $\lambda_2 = \frac{2\pi}{\kappa_2}$). The field at point X originates in inhomogenieties of the index of refraction at x having the spatial wave number $\kappa_2\,\frac{X}{x}$ (or dimension $\lambda_2 = \lambda_2\,\frac{x}{X}$). This is a classical result (see Tatarski [14]).

From the definitions of $\kappa_2, \kappa, \lambda_2'$ it is seen that

$$\frac{X(X-x)\kappa^2}{2kx} = \pi\,\lambda x\,\frac{(X-x)/X}{\lambda_2^2}$$

$$\qquad'' \qquad = \frac{\text{square of radius of 1st Fresnel zone}}{\text{square of size of inhomogeneity}}$$

Second moment of the random wave field

The random wave field $\psi(X,\rho)$ has a second moment Γ given by

$$\Gamma = \langle \psi_1(X, R_1)\,\psi_1^*(X+\varsigma, R_2)\rangle \quad (20.16)$$

$$\varsigma = (Y'-Y'')\hat{j} + (Z'-Z'')\hat{k}$$

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Thus we obtain a quadruple integral over K_2', K_3', K_2'', K_3'', viz.

$$\Gamma(X, \underline{\varrho}; R_1, R_2) = \iiiint_{-\infty}^{\infty} \exp\left[i\left(K_2' Y' - K_2'' Y'' + K_3' Z' - K_3'' Z''\right)\right]$$
$$\times \langle d\Phi(X, K_2', K_3'; k_1) d\Phi^*(X, K_2'', K_3''; k_2)\rangle \tag{20.17}$$

and

$$\langle d\Phi\, d\Phi^*\rangle = \frac{a_0^2 k_1 k_2\, 4\pi^4}{X^2} \iint dx_1\, dx_2 \exp\left[-i(H' - H'')\right]$$
$$\langle dN(x_1, \tfrac{X}{x_1} K_2', \tfrac{X}{x_1} K_3')\, dN^*(x_2, \tfrac{X}{x_2} K_2'', \tfrac{X}{x_2} K_3'')\rangle$$
$$H' = \frac{X(X-x)}{2k_1 x}\left(K_2'^2 + K_3'^2\right) \quad; \quad H'' = \frac{X(X-x)\left(K_2''^2 + K_3''^2\right)}{2k_2 x_2}.$$

It is now assumed that in the wave field the spatial components $\frac{X}{x_1} K_2'$ and $\frac{X}{x_2} K_2''$, etc. at distance X are uncorrelated in the YZ plane. Thus,

$$\langle dN\, dN^*\rangle = \langle \epsilon^2\rangle\, \delta\left(\frac{X}{x_1} K_2' - \frac{X}{x_2} K_2''\right) \delta\left(\frac{X}{x_1} K_3' - \frac{X}{x_2} K_3''\right)$$
$$F\left(|x_1 - x_2|, \tfrac{X}{x_1} K_2', \tfrac{X}{x_1} K_3'\right) \frac{X^4}{x_1^2 x_2^2} dK_2'\, dK_2''\, dK_3'\, dK_3'' \tag{20.18}$$

in which F is the 2-dimensional Fourier spectrum of the correlation function of the random index of refraction. Since

$$\delta\left(\frac{X}{x_1} K_2' - \frac{X}{x_2} K_2''\right) = \delta\left(K_2'' - \frac{x_2}{x_1} K_2'\right) \frac{x_2}{X}$$

integration over K_2'', K_3'' leads to the result that

$$\Gamma(X, \underline{\varrho}; k_1, k_2) = a_0^2 k_1 k_2\, 4\pi^4 \langle \epsilon^2\rangle \iint \frac{dx_1\, dx_2}{x_1^2}$$
$$\times \iint dK_2'\, dK_3'\, F_n\left(|x_1 - x_2|, \tfrac{X}{x_1} K_2', \tfrac{X}{x_1} K_3'\right) \tag{20.19}$$
$$\exp i\left[K_2'\left(Y' - \tfrac{x_2}{x_1} Y''\right) + K_3'\left(Z' - \tfrac{x_2}{x_1} Z''\right)\right]$$
$$\exp -i\left[\frac{X}{2x_1}\left\{\frac{X - x_1}{k_1} - \frac{x_2}{x_1 k_2}(X - x_2)\right\}\left(K_2'^2 + K_3'^2\right)\right]$$

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To simplify, change variables as follows

$$x_1 = \eta \quad ; \quad x_2 = \eta - \xi$$

The absolute value of the Jacobian of transformation is 1. Hence

$$\Gamma(X, \underline{\varrho} ; k_1, k_2) = a_0^2 k_1 k_2 \, 4\pi^4 \langle \epsilon^2 \rangle \int_0^X \frac{d\eta}{\eta^2} \int_\eta^{\eta-X} d\xi \iint_{-\infty}^\infty dK_2' dK_3'$$

$$F_m\left(|\xi|, \frac{X}{\eta} K_2, \frac{X}{\eta} K_3\right) \exp i\left[K_2'\left(Y' - \frac{\eta-\xi}{\eta} Y''\right) + K_3'\left(Z' - \frac{\eta-\xi}{\eta} Z'\right)\right]$$

$$\times \exp i\left[-\frac{X}{2\eta}\left\{\frac{X-\eta}{k_1} - \frac{(\eta-\xi)}{\eta k_2}(X - \eta + \xi)\right\}(K_2'^2 + K_3'^2)\right] \tag{20.20}$$

Now in the symbolic form $F_m(\underline{\xi}, \underline{\kappa})$ where $K = \sqrt{K_2^2 + K_3^2} = \frac{2\pi}{L}$, it is seen from physical argument that if the size of the inhomogeniety (ℓ) is small the function F_m describing the spatial coherence of inhomogeneities vanishes for large distances $\underline{\xi}$, that is F_m contributes appreciable values only if $K\underline{\xi}$ is small ($\lesssim 1$). Following Tatarski [14] one sets the requirement that

$$\frac{X}{\eta} K' |\xi| \lesssim 1 \tag{20.21}$$

Since the basic stipulation in the wave scattering process is that the distance X to the observation point be much larger than the size of the inhomogeneities that is

$$K' \underline{X} \gg 1 \tag{20.22}$$

the two requirements (20.21) and (20.22), taken together, lead to the requirement that

$$|\underline{\xi}| \ll \eta \tag{20.23}$$

The physical significance of this is that in the integration process over the volume inhomogeneities the contribution to the scattering from a volume element at a point η is restricted only to the immediate neighborhood of η . Furthermore, F_m falls off rapidly to zero with $|\underline{\xi}|$ even under (20.23). Setting the limits of integration at $\pm\infty$ and noting that

$$\Phi_m(K_1, K_2, K_3) = \frac{1}{2\pi} \int_{-\infty}^\infty F_m(\underline{\xi}, k_2, k_3) \cos(K_1 \underline{\xi}) d\underline{\xi}$$

one finally arrives at the form

$$\tag{20.24}$$

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$$\Gamma(X, \underline{\varrho}; \dot{k}_1, \dot{k}_2) = a_0^2 k_1 k_2 \, 4\pi^4 \langle \epsilon^2 \rangle \int_0^X \frac{d\eta}{\eta^2} \iint dK_2' \, dK_3'$$

$$\Phi_n \left(0, \frac{X}{\eta} K_2', \frac{X}{\eta} K_3'\right) exp \, i \left[\left\{ K_2'(Y'-Y'') + K_3'(Z'-Z'')\right\} - \frac{X}{2\eta}\right.$$

$$\left. \cdot (X-\eta)\left(\frac{1}{k_1} - \frac{1}{k_2}\right)\left(K_2'^2 + K_3'^2\right) \right. \tag{20.25}$$

21. The Markov Approximation

For time exp $(-i\omega t)$ the parabolic equation describing the diffusion of a wave in a random medium has the form

$$-2ik \frac{\partial u}{\partial x}(x, \underline{\varrho}) + \nabla_\perp^2 u = -k^2 \epsilon (x, \underline{\varrho}) u(x, \underline{\varrho}) \tag{21.1}$$

In an unbounded medium the Green's function $g(x, \underline{\varrho}; x', \underline{\varrho}')$ of the diffusion equation for $\epsilon = 0$ is known to be

$$g(x, \underline{\varrho}; x', \underline{\varrho}') = \frac{ik}{2\pi(x-x')} exp \left\{ \frac{ik(\underline{\varrho}-\underline{\varrho}')^2}{2(x-x')} \right\} H(x-x') \tag{21.2}$$

in which $H(x-x')$ is the Heaviside step function,

$$
\begin{array}{lll}
H(x-x') = 1 & & x > x' \\
\quad '' \quad = 0 & & x < x' \\
\quad '' \quad = \frac{1}{2} & & x = x'
\end{array}
\tag{21.3}
$$

We next suppose there is a true source function $\delta(\underline{\varrho} - \underline{\varrho}')$, and seek a Green's function G such that

$$\frac{\partial G}{\partial x} + \frac{1}{2ik} \nabla^2 G = \frac{ik}{2} \epsilon G + \delta(\underline{\varrho} - \underline{\varrho}') \tag{21.4}$$

Since (21.2) is a solution when $\epsilon = 0$, it is seen that

$$G(x, \underline{\varrho}; x', \underline{\varrho}') = g(x, \underline{\varrho}; x', \underline{\varrho}') + \frac{ik}{2} \int_{x'}^{x} d\xi \int d\underline{r} \tag{21.5}$$

$$\cdot g(x, \underline{\varrho}; \xi, \underline{r}) \, \epsilon(\xi, \underline{r}) \, G(\xi, \underline{r}; x', \underline{\varrho}')$$

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Thus, when $\epsilon \neq 0$ the free field Green's function g is changed to G by the addition of a quantity proportional to ϵ and dependent on G itself. We return to (21.1) and write a solution in the form of a sum of complementary and particular solutions. Taking the ensemble average of the solution, one obtains:

$$\langle u(x, \underline{\varsigma}) \rangle = u^0(\underline{\varsigma}) \exp \langle \{ - ik \int_0^x d\xi \, \epsilon(\xi, \underline{\varsigma}) \} \rangle$$

$$+ \langle \frac{ik}{2} \int_0^x d\xi \exp \{ \frac{ik}{2} \int_\xi^x d\eta \, \epsilon(\eta, \underline{\varsigma}) \} \nabla_\perp^2 u(\xi, \underline{\varsigma}) \rangle \tag{21.6}$$

Since $u(\xi, \underline{\varsigma})$ is a functional of $\epsilon(\xi, \underline{\varsigma})$, it is correlated to $\epsilon(\eta, \underline{\varsigma})$. The strength of this correlation depends on the ratio of the correlation length of ϵ and the correlation length of the component of u, in the direction of propagation. For long path lengths this ratio may be assumed small. In particular, using the casuality principal, if the wave u is at $\xi, \underline{\varsigma}$, we assume it is uncorrelated to the random ϵ at $\eta, \underline{\varsigma}$ when $\eta > \xi$. Thus, when $\eta > \xi$ the ensemble average of the product $u\epsilon$ becomes

$$\langle u\epsilon \rangle = \langle u \rangle \langle \xi \rangle \tag{21.7}$$

that is, u and ϵ are statistically independent for all $\eta > \xi$. Eq. (21.7) is a basic condition of a Markov random process. Thus, in the Markov approximation

$$\langle u(x, \underline{\varsigma}) \rangle = u^0(\underline{\varsigma}) \langle \exp \{ \frac{-ik}{2} \int_0^x d\xi \, \epsilon(\xi, \underline{\varsigma}) \} \rangle + \frac{ik}{2} \int_0^x d\xi$$

$$\times \langle \exp \{ -\frac{ik}{2} \int_\xi^x d\eta \, \epsilon(\eta, \underline{\varsigma}) \rangle \nabla_\perp^2 \langle u(\xi, \underline{\varsigma}) \rangle \tag{21.8}$$

This equation can also be derived by assuming the correlation function R_ϵ for the field ϵ to be delta correlated in the direction of propagation,

$$R_\epsilon(x, x'; \underline{\varsigma}, \underline{\varsigma}') = \delta(x - x') A_\epsilon(\underline{\varsigma} - \underline{\varsigma}') \tag{21.9}$$

(Tatarski [JETP 29, 1130 (1969)]). If ϵ is Gaussian, or if the phase function $S(x, \underline{\varsigma})$ is Gaussian where,

$$S(x, \underline{\varsigma}) = \frac{k}{2} \int_0^x d\xi \, \epsilon(\xi, \underline{\varsigma}) \tag{21.10}$$

then Eq. (21.8) is equivalent to the d.e.

$$-2ik \frac{\partial \langle U \rangle}{\partial x} + \nabla_{\perp}^2 \langle U \rangle = \frac{ik^3}{4} A_{\epsilon}(0) \langle U \rangle \tag{21.11}$$

the solution of which is

$$\langle U(x, \underline{\varsigma}) \rangle = U^0(x, \underline{\varsigma}) \, exp\left(-\frac{x}{2d}\right) \tag{21.12}$$

in which d is the extinction distance in the (first) Born approximation.

The d.e. equations for the higher order mements of U can also be found in an analogous way. The second moment (or mutual coherence function $\langle \Gamma_2 \rangle$) is governed by the relation

$$-2ik \frac{\partial \Gamma_2}{\partial x} + (\nabla_{\perp_1}^2 - \nabla_{\perp_2}^2) \Gamma_2 = \frac{ik^3}{2} \left[A(0) - A(\underline{\varsigma}_1 - \underline{\varsigma}_2) \right] \Gamma_2 \tag{21.13}$$

or equivalently by the integral equation

$$\langle \Gamma_2(x; \underline{\varsigma}_i, \underline{\varsigma}_i') \rangle = \Gamma_2^0(\underline{\varsigma}, \underline{\varsigma}_i') \langle exp\left\{ -\frac{ik}{2} \int_0^x d\xi \left[\epsilon(\xi, \underline{\varsigma}) - \epsilon(\xi, \underline{\varsigma}') \right] \right\} \rangle$$

$$+ \frac{i}{2k} \int_0^x d\xi \, \langle exp\left\{ \frac{ik}{2} \int_{\xi}^{\eta} d\eta \left[\epsilon(\eta, \underline{\varsigma}) - \epsilon(\eta, \underline{\varsigma}') \right] \right\} (\nabla_{\perp\underline{\varsigma}_i}^2 - \nabla_{\perp\underline{\varsigma}_i'}^2)$$

$$\times \langle \Gamma_2(\xi; \underline{\varsigma}_i, \underline{\varsigma}_i') \rangle. \tag{21.14}$$

(Klyatskin [JETP 30, 520 (1970)])

The fourth-order moment Γ_4 that determines the intensity fluctuation, viz.

$$\Gamma_4(x; \underline{R}, \underline{r}_1 | \underline{r}_2, \underline{\varsigma}) \equiv \langle U\left(x, \underline{R} + \frac{\underline{r}_1 + \underline{r}_2}{2} + \frac{\underline{\varsigma}}{4}\right) U\left(x, \underline{R} - \frac{\underline{r}_1 + \underline{r}_2}{2} + \frac{\underline{\varsigma}}{4}\right)$$

$$U^*\left(x, \underline{R} + \frac{\underline{r}_1 - \underline{r}_2}{2} - \frac{\underline{\varsigma}}{4}\right) U^*\left(\underline{x}, R - \frac{\underline{r}_1 - \underline{r}_2}{2} - \frac{\underline{\varsigma}}{4}\right) \rangle \tag{21.1}$$

is governed by the d.e.

$$\frac{\partial \Gamma_4}{\partial x} - \frac{i}{4} \left(\nabla_{\perp\underline{R}}^2 \nabla_{\perp\underline{\varsigma}}^2 + \nabla_{\perp\underline{r}_1}^2 \nabla_{\perp\underline{r}_2}^2 \right) \Gamma_4 = \frac{\pi k^2}{4} F(\underline{r}_1, \underline{r}_2, \underline{\varsigma}) \Gamma_4 \tag{21.16}$$

where

$$F(\underline{r}_1, \underline{r}_2) = H\left(\underline{r}_1 + \frac{\underline{\varsigma}}{2}\right) + H\left(\underline{r}_1 - \frac{\underline{\varsigma}}{2}\right) + H\left(\underline{r}_2 + \frac{\underline{\varsigma}}{2}\right) + H\left(\underline{r}_2 - \frac{\underline{\varsigma}}{2}\right)$$

$$- H(\underline{r}_1 + \underline{r}_2) - H(\underline{r}_1 - \underline{r}_2) \tag{21.17}$$

$$H(\underline{\varsigma}) = \frac{1}{\pi}\left[A(0) - A(\underline{\varsigma})\right] = 2\iint\limits_{-\infty}^{\infty}(1 - \cos\underline{\varkappa}\cdot\underline{\varsigma})\,\Phi_\epsilon\,(0,\underline{\varkappa})\,d^2\underline{\varkappa} \tag{21.18}$$

(due to Tartarki as quoted in Barabanenkov [USPEKHI 13 551 (1971)]) (Ref. 15).

The solution for Γ_2 in Eq. (21.11) may be obtained by a Fourier transformation over $\underline{R} = \frac{1}{2}(\underline{\varsigma}_1 + \underline{\varsigma}_2)$. It is.

$$\Gamma_2\,(x,\underline{R},\underline{\varsigma}) = \langle U(x,\underline{R}+\tfrac{1}{2}\underline{\varsigma})\,U^*(x,\underline{R}-\tfrac{1}{2}\underline{\varsigma})\rangle$$

$$= \frac{1}{4\pi^2}\iint\limits_{-\infty}^{\infty}d^2\underline{R}'\iint\limits_{-\infty}^{\infty}d^2\underline{\varkappa}\,\exp\left\{i\underline{\varkappa}\cdot(\underline{R}-\underline{R}') - \frac{\pi k^2}{4}\int_0^x H(\underline{\varsigma} - \frac{\underline{\varkappa}(x-\xi)}{k})\,d\xi\right\}$$

$$\times\langle U(0;\underline{R}',\underline{\varsigma} - \frac{x\underline{\varkappa}}{k})\,U^*(0;\underline{R}',\underline{\varsigma} - \frac{x}{k}\underline{\varkappa})\rangle \tag{21.19}$$

(due to Dolin as quoted in Barabanenkov). Thus, one may calculate the decay of mutual coherence between two points $\underline{R},\underline{\varsigma}$ in a transverse plane with propagation distance x if the mutual coherence function in the plane is known and if the randomness of the index of refraction has a known power density spectrum or a known structure function (or correlation function).

The solution for the fourth moment Γ_4 (describing the fluctuations in intensity in transverse planes in the direction of propagation) may be obtained by restricting the scattering to the small angle approximation. Letting ϕ be a double Fourier transform of Γ_4 in the pairs $\underline{p}\cdot\underline{R}$ and $\underline{\varkappa}\cdot\underline{r}_2$; i.e.,

$$\Gamma_4\,(x;\underline{R},\underline{r}_1,\underline{r}_2,\underline{\varsigma}) = \iint\limits_{-\infty}^{\infty}\exp(i\,\underline{p}\cdot\underline{R})\,d^2\underline{p}\iint\limits_{-\infty}^{\infty}e^{i\underline{\varkappa}\cdot\underline{r}_2}\,d^2\underline{\varkappa} \tag{21.20}$$

$$\times\,\varphi\,(x;\underline{p},\underline{r}_1,\underline{\varkappa},\underline{\varsigma} - \underline{p}\frac{x}{k})$$

and substituting this form of Γ_4 into (21.16) one obtains the following integral-differential equation in φ:

$$\frac{\partial\varphi}{\partial x} + \frac{\underline{\varkappa}}{k}\cdot\nabla_{\underline{r}_1}\varphi + \frac{\pi k^2}{4}\left[H\,(\underline{r}_1+\frac{\underline{\varsigma}}{2}) + H\,(\underline{r}_1 - \frac{\underline{\varsigma}}{2})\right]\varphi$$

$$= \pi k^2\iint\limits_{-\infty}^{\infty}\Phi_\epsilon\,(\underline{\varkappa}')\left[\cos\frac{\underline{\varkappa}\cdot\underline{\varsigma}}{2} - \cos\underline{\varkappa}'\cdot\underline{r}_1\right]\varphi\,(x;\underline{p},\underline{r}_1,\underline{\varkappa}-\underline{\varkappa}',\underline{\varsigma})\,d^2\underline{\varkappa}'$$

$$\tag{21.21}$$

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(Due to Klyatskin and Tatarski [Radiofisika 13,7,828 (1970)]. This is a radiation-transport equation. It may be solved approximately by replacing φ within the integral of the r.h.s. by the known initial value $\varphi = \varphi(0; \underline{\rho}, \underline{z}, \underline{x}-\underline{x}')$. The distribution of Γ_4 is generally not Gaussian. Hence, U is generally not Gaussian.

The limits of applicability of the Markov approximation are the same as for the parabolic equation. They have been elaborated by Tatarski and Klyatskin [JETP 31,332 (1970)] and are listed here:

$$(a)\ \ \langle \tilde{\epsilon}^2 \rangle k \ell \ll 1, \quad \text{or} \quad \lambda \alpha \ll 1, \quad \alpha \sim \langle \tilde{\epsilon}^2 \rangle k^2 \ell$$

$$(b)\ \ \langle \tilde{\epsilon}^2 \rangle \frac{x}{\ell} \ll 1$$

$$(c)\ \ \langle \tilde{\epsilon}^2 \rangle kx \ll 1$$

(21.22)

Condition (a) states that the attenuation per unit wavelength (namely α) must be very small. Condition (b) states that the mean-square of the fluctuations in the direction of propagation must be small. Condition (c) states that the effect of backward reflected waves must be negligible. To these must be added the conditions of applicability of the parabolic equation,

$$(d)\ \ \lambda \ll \ell \qquad\qquad (e)\ \ \frac{\lambda x}{\ell^2} \ll \frac{\ell^2}{\lambda^2}$$

(21.23)

Further Discussion on the Applicability of the Parabolic Equation

Let λ be the wavelength, ℓ the scale of inhomogeneity, L the path length, $\epsilon(\underline{\lambda},t)$ the random component of the index of refraction, and k the wavenumber (consider a constant). It is well known [Tatarski, loc. cit. p. 223] that the partial differential equation;

$$\frac{\partial u}{\partial y^2} + \frac{\partial u}{\partial z^2} + 2ik \frac{\partial u}{\partial x} = -k^2 \epsilon(\underline{x})$$

(21.24)

has the exact solution

$$u(L, y, z) = \frac{k^2}{4\pi} \int_0^L dx' \iint_{-\infty}^{\infty} dy'\,dz'\ \frac{\exp\left\{ik\ \frac{(y-y')^2+(z-z')^2}{2(L-x')}\right\}}{L-x'}\ \epsilon(x', y', z')$$

(21.25)

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Now the exact kernel of the Green's function for the Helmholtz equation is $\exp\{ik|\underline{z}-\underline{z}'|\}/|\underline{z}-\underline{z}'|$ which reduces to the kernel $\exp\left\{ik\frac{(y-y')^2+(z-z')^2}{2(L-x')}\right\}/(L-x')$ when

$$(1) \quad \lambda^3 L \ll \ell^4 \tag{21.26}$$

This is a first condition for the applicability of the parabolic equation in the propagation of sound in a random medium. A second condition is based on the dropping of the term $\partial^2 u/\partial x^2$. It is seen that if u varies significantly over a scale size ℓ then $\partial^2 u/\partial x^2$ is of the order u/ℓ^2 while $2ik\,\partial u/\partial x$ is of the order $u/\lambda\ell$. Thus one requires a second condition,

$$(2) \quad \lambda \ll \ell \tag{21.27}$$

A third condition arises from the fact that the parabolic equation is first order in x, and thus ignores the back-scattered (or reflected) wave. Since reflection is a function of the scale of inhomogeneity it is convenient to measure scale with reference to the smallest size ℓ_0 and the largest size L_0. Applying Kolmogorov's theory of turbulence it can be shown that the condition on the wavelength is

$$(3) \quad \lambda^2 \ll \ell_0 \left(\frac{\ell_0}{L_0}\right)^{5/3} \tag{21.28}$$

and the (simultaneous) condition on the path length is

$$(4) \quad k^2 C_\epsilon^4 L_0^{5/3} \ell_0^{-\frac{1}{3}} L^2 \ll 1 \tag{21.29}$$

in which C_ϵ is the Kolmogorov structure function constant [see Tatarski, loc. cit. p. 409].

A fifth condition is associated with the attenuation of the mean field over distances of the order of a wavelength. Since the scattering of the wave must only be forward it is required that the acoustic size of the smallest inhomogeneity be large,

$$(5) \quad k\ell_0 \gg 1 \tag{21.30}$$

and the attenuation α per wavelength be small,

$$(6) \quad \lambda\alpha \ll 1 \tag{21.31}$$

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These conditions supplement the discussions of the previous section.

22. The Iteration Procedure of Beran [J. Opt. Soc. Am. 56, 1475 (1966)]

The propagation of the mutual coherence function $\langle \Gamma(\underline{x}_1, \underline{x}_2, \tau) \rangle$ is governed by an equation of the form

$$\mathscr{L}_1 \mathscr{L}_2 \langle \Gamma(\underline{x}_1, \underline{x}_2, \tau) \rangle = \langle \tilde{n}^2(\underline{x}_1) \tilde{n}^2(\underline{x}_2) \mathscr{L}_3 \Gamma(\underline{x}_1, \underline{x}_2, \tau) \rangle$$

$$\mathscr{L}_{1,2} \equiv \nabla_{1,2}^2 - \frac{n_o^2}{c^2} \partial^2/\partial\tau^2$$

$$\mathscr{L}_3 \equiv \frac{1}{c^2} \partial^4/\partial\tau^4 \tag{22.1}$$

in which the index of refraction squared has the form

$$n^2(\underline{x}) = n_o^2 + \tilde{n}^2(\underline{x}) \tag{22.2}$$

It is required to solve (22.1) over a path length L. One method (Beran) is that of iteration.

The propagation length in the direction $\underline{z}$ is divided into M slabs, $\Delta z = L/M$, where Δz is small enough, such that

$$\langle \tilde{n}^2(\underline{x}_1) \tilde{n}^2(\underline{x}_2) \Gamma(\underline{x}_1, \underline{x}_2, \tau) \rangle \approx \langle \tilde{n}^2(\underline{x}_1) \tilde{n}^2(\underline{x}_2) \rangle \langle \Gamma(\underline{x}_1, \underline{x}_2, \tau) \rangle \tag{22.3}$$

This means in each slab the local field that correlates with $\tilde{n}(\underline{x})$ (and thus generates scattering) is constant along the path Δz and is equal to the field at the initial boundary of the slab; i.e., it is assumed that the stochastic wave and the stochastic index of refraction are locally independent. Thus, the known initial field at boundary $z=0$ provides the the value of Γ_0 to be inserted on the r.h.s. of (22.1) as applied to the first slab. Subject to (22.3), Eq. (22.1) is then solved for the field $\Gamma_{\Delta z}$. This field, in turn, is inserted into Eq. (22.1) and (22.3) and applied to the second slab, where a solution is obtained. In this way, the field $\Gamma(\underline{x}_1, \underline{x}_2, \tau)$ at L is obtained after M iterations. The scattered portion Γ_s of the solution of (22.1) subject to (22.3) is obtained by using the free space (deterministic) Green's function. The result (in the jth slab) is,

$$\langle \hat{\Gamma}_s(\underline{x}_1, \underline{x}_2, \nu) \rangle = \frac{k^4}{(4\pi)^2} \int_{V_1'} d^3x_1' \int_{V_2'} d^3x_2' \frac{\exp\{-ik[|x_1-x_1'| - |x_2-x_2'|}{|\underline{x}_1 - \underline{x}_1'| |\underline{x}_2 - \underline{x}_2'|}$$

$$\langle \tilde{n}_2^2(\underline{x}_1) \tilde{n}_2^2(\underline{x}_2) \rangle \langle \Gamma_{j\Delta z}(\underline{x}_1', \underline{x}_2', \nu) \rangle \tag{22.4}$$

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The Green's function is next simplified to the Fresnel approximation, so that only small angle scattering is assumed. Thus modified, Eq. (22.4) is solved by well-known methods. The result after M iterations is:

$$\langle \hat{\Gamma}_{M\Delta z} \left(X_{12}, Y_{12}, 0, L, \nu \right) \rangle = \langle \hat{\Gamma}_0 \left(X_{12}, Y_{12}, 0, 0, \nu \right) \rangle$$

$$\exp\left\{ k^2 L \left[\bar{\sigma} \left(x_{12}, y_{12} \right) - \bar{\sigma} \left(0, 0 \right) \right] \right\} \tag{22.5}$$

where

$$\bar{\sigma} \left(x_{12}, y_{12} \right) = \frac{1}{4} \int_{-\infty}^{\infty} d\Delta_z \, \sigma \left(x_{12}, y_{12}, \Delta_z \right)$$

$$\underline{\Delta} = \underline{x}_2' - \underline{x}_1' = \Delta \left(\Delta x, \Delta_y, \Delta_z \right)$$

$$\underline{X}_{12} = \underline{x}_2 - \underline{x}_1 = \left(x_{12}, y_{12}, z_{12} \right)$$

$$\sigma_{12} \left(\underline{X}_{12} \right) \equiv \langle \tilde{n}^2 \left(\underline{x}_1 \right) \tilde{n}^2 \left(\underline{x}_2 \right) \rangle$$

The validity of the procedure is narrowly limited by the following conditions:

$$\frac{(\Delta z) k^2}{4} \int_{-\infty}^{\infty} \sigma \left(0, 0, z_{12} \right) d z_{12} \ll 1 \tag{22.6}$$

'Fresnel' Green's function: $\Delta z \ll \frac{\ell^4}{\lambda^3}$, ℓ = smallest scale of the inhomogeneities

Max. scale of inhomogeneity: $\ell_M / \Delta z \ll 1$

Extreme scales of inhomogeneity: $\ell / \lambda \gg 1$; $\lambda \ell_M \ell^2 \ll 1$

The requirement $\ell_M \ll \Delta z$ means the slab thickness must be many times larger than the largest scale of inhomogeneity.

23. Procedure for Solution of Reflection and Transmission Coefficients of Random Medium. [Lang, J. Math. Phys. 14, 1921 (1973)].

A scalar wave normally incident on a one dimensional slab of random medium is reflected and transmitted. It is required to find the random reflection and transmission coefficients, and the statistical moments of the random wave field inside the medium.

The first step in solving this problem is to convert the two-point boundary value problem of the field inside the slab into two cascaded initial value problems: the first (problem) in the reflection coefficient $\Gamma = \rho e^{i\phi}$; and the second (problem) in the wave field itself $u = e^{\gamma + i\theta}$. This is done by the technique of invariant embedding. Since the index of refraction ($= \mu$) is random, the four functions $\rho, \phi, \gamma, \theta$ are also random.

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The first initial value problem (in the reflection coefficient Γ) is solved by the following procedure:(see Sect. 12 for procedures):

(1) The scalar random index of refraction μ is assumed to be a zero mean, exponential correlated gaussian process generated by an $\hat{I}$ to stochastic differential equation in $d\mu$, driven by white noise $w = d\beta/d\varkappa$, where β is a Brownian motion process. Thus $d\mu$ itself is a gaussian-markov process.

(2) The state variable scalar $d\mu$ is augmented by the variable $d\Gamma = f\left(\Gamma, \mu, \varkappa\right)d\varkappa$, obtained from the first initial value problem, to form a vector gaussian-markov process $\underline{X} = \left(\varsigma, \phi, \mu\right)$ which satisfies a set of stochastic differential equations of the form

$$d\underline{X} = \underline{f}\left(\underline{X}, \varkappa\right)d\varkappa + \underline{\underline{G}}\, d\underline{\beta} \tag{23.1}$$

(3) The probability density function $p_1\left(\underline{X}, \varkappa\right)$ is known to obey the forward Kolmogorov differential equation with a prescribed initial condition, whose form depends on $\underline{f}$ and $\underline{\underline{G}}$. This equation is solved by perturbation, and the solution $p_1\left(\underline{X}, \varkappa\right)$ is integrated over μ to give $p_1\left(\hat{\underline{X}}, \varkappa\right)$, where $\hat{\underline{X}} = \left(\varsigma, \phi\right)$.

(4) The statistical moments of Γ are found by integration over $d\hat{\underline{X}}$ of the solved p.d.f. $p_1\left(\hat{\underline{X}}, \varkappa\right)$ with $\varkappa = L =$ thickness of the slab.

The second initial value problem (in the field u) is solved by the (analogous) procedure:

(1) The scalar index of refraction μ is again assumed to be generated by a (spatial) Wiener-Lévy process (=Brownian motion). Specifications on μ are the same as (1) above.

(2) The state variable $d\mu$ is augmented by both $d\Gamma$ from the first initial value problem and $du = f\left[u, \varkappa, \Gamma, \mu\right]d\varkappa$ from the second initial value problem. The augmented state variable is a five-dimension vector $\underline{Y} = \left(v, \theta, \varsigma, \phi, \mu\right)$ which is governed by the vector gaussian-markov stochastic differential equation,

$$d\underline{Y} = \underline{f}\left[\underline{Y}, \varkappa\right]d\varkappa + \underline{\underline{G}}\, d\underline{\beta} \tag{23.2}$$

(3) The probability density function $p_2\left(\underline{Y}, \varkappa\right)$ is known to obey the forward Kolmogorov equation with prescribed initial conditions, whose form depends on $\underline{f}$ and $\underline{\underline{G}}$. This equation is solved by perturbation and the solution is integrated over μ to give $p_2\left(\underline{Y}, \varkappa\right)$, where $\hat{\underline{Y}} = \left(v, \theta, \varsigma, \phi\right)$.

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(4) The statistical moments of $u(x, z)$ are obtained by integration over dy of the solved $\xi_a(y, v)$ with $x = L =$ thickness of the slab.

24. Mode Coupling in Random Waveguides (G.C. Papanicolaou, J. Math Phys. 13, p. 1915, 1972)

Let $u(\underline{x})$ be a time harmonic complex field in a wave guide having a propagation wave number $k_n z$ where k is the free space number and $n(\underline{x})$ the index of refraction, taken as a stationary random field that deviates little from its mean value. Assume the wave guide to have a transverse area A (y, z) bounded by a curve $S(y, z)$, and extending in the x-direction from $-\infty$ to $+\infty$. Let $f_m(y, z)$ be the eigenfunctions of the wave operator in A i.e., $\partial_{yy} + \partial_{zz} = -\lambda_m^2 f_m$, $m = 0, 1, 2, \ldots$, and satisfying either Dirichlet or Neumann conditions on S. Thus, in general, $\nabla^2 u + k^2 n^2 u = 0$, and $u = \sum_n B_n f_n$.

The first step in the application of the method of Keller-Papanicolaou to the boundary value problem for $u(\underline{x})$ is to convert it to an initial value problem. This is done by restricting the <u>scattering to be forward only</u>, using the parabolic (i.e. "quasi optics") approximation. The field $u(\underline{x})$ is written $\exp(ikx) v(\underline{x})$, and then $\partial_{xx} u$ is neglected, leading to an equation in $v(\underline{x})$ of the form

$$(\partial_{yy} + \partial_{zz}) v + 2ik \partial_x v + k^2 (n^2 - 1) v = 0 \tag{24.1}$$

Writing $n^2 - 1 = \frac{\epsilon}{k} \mu(\underline{x})$, where μ is a stationary zero mean random field, one arrives at a first order equation (in x) for $v(x)$, i.e.

$$2i \partial_x v(\underline{x}) = -\epsilon \mu v(\underline{x}) - (\partial_{yy} + \partial_{zz}) v(\underline{x}) , \quad x > 0 \tag{24.2a}$$

$$v(0, y, z) = v_0(y, z) \tag{24.2b}$$

This is an initial value problem in which $v(\underline{x})$ satisfies the same boundary conditions on S as $u(\underline{x})$. The function $v(\underline{x})$ i next expanded in eigenfunctions $v(\underline{x}) = \sum_m v_m(x) f_m(y, z)$ (24.3) Substituting (24.3) into (24.2a), and using orthogonality of f_m , one arrives at an equation for each $v_p(x)$,

$$\frac{dv_p(x)}{dx} = i k_p v_p(x) + \epsilon \sum_q i \mu_{pq}(x) v_q(x) \tag{24.4a}$$

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$$v(0, y, z) = v_p(0), \qquad p = 0, 1, 2, \cdots \cdots \tag{24.4b}$$

in which $v_p(0)$ is given. Here

$$k_p = -\frac{1}{2}\lambda_p^2 \; ; \quad \mu_{pq}(x) = \frac{1}{2}\int_A \mu(x,y,z)\, h_q(y,z)\, h_p(y,z)\, dy\, dz$$

$$\int h_p(y,z)\, h_q(y,z) = \delta_{pq} \tag{24.4c}$$

Equations (24.4) represent an initial value problem in $v_p(x)$ for each p. Choosing the first n of these equations and noting that $p, q = 0, 1, 2, \ldots n$, one can write the finite order <u>vector stochastic initial value problem</u> in the n-vector $\underline{V}(x)$ of the scalar x,

$$\frac{d\underline{V}(x)}{dx} = \left(i\underline{k} + \epsilon \underline{\underline{X}}(x)\right)\underline{V}(x) = \left(v_1, v_2, \cdots v_n\right)^T \tag{24.5}$$

$$\underline{V}(0) = \underline{u}_0 \; ; \quad \underline{\underline{X}}(x) = i\mu_{pq}(x), \quad \underline{k} = diag(k_1, k_2, \cdots k_n)$$

In sum: $\underline{V}(x)$ is a complex n-vector function representing modal amplitudes of the field; $\underline{k}$ is a diagonal matrix of eigenvalues of k_p ; $\underline{\underline{X}}(x)$ is a (generally) complex pq-matrix valued stochastic process which describes the coupling of the inhomogeneities $\mu(x)$ to the wave guide modes; and ϵ is a small parameter.

The process $\mu_{pq}(x)$ is a random function of scalar x. It is assumed that μ is a zero mean, wide-sense stationary process: i.e.,

(1) $\quad \langle \mu_{pq}(x)\rangle = 0$ \hfill (24.6a)

(2) $\quad \langle \mu_{pq}(x+x')\,\mu_{p'q'}(x)\rangle = R_{pq,p'q'}(x')$ \hfill (24.6b)

To solve Eq. (24.5) it is convenient first to write $\underline{V}(x) = e^{i\underline{k}x}\,\underline{U}(x)$, i.e. the field is assumed oscillatory with a slowly varying amplitudes. Then

$$\frac{\partial \underline{U}(x)}{\partial x} = \epsilon \underline{\underline{\tilde{X}}}\,\underline{U}(x), \quad \underline{U}(0) = \underline{u}_0 \;(given) \tag{24.7}$$

$$\underline{\underline{\tilde{X}}} = e^{-ikx}\,\underline{\underline{X}}(x)\, e^{ikx}$$

Now let $\underline{\underline{y}}(x)$ be the tensor product $\underline{U}\otimes\underline{U}$, that is, let the elements of $\underline{\underline{y}}$ be the dyadic $U_p U_p^*$. Differentiating $\underline{\underline{y}}$ and using (24.7) one arrives at the form,

$$\frac{d\underline{y}(x)}{dx} = \epsilon \, \underline{\underline{V}}(x)\underline{y}(x) \, , \qquad \underline{c}_0 = \underline{u}_0 \otimes \underline{u}_0^* \tag{24.8}$$

$$\underline{\underline{V}}(x) = \tilde{\underline{X}} \otimes \tilde{\underline{I}} + \tilde{\underline{I}} \otimes \tilde{\underline{X}}^*(x)$$

which has the components,

$$d\underline{y}_{pp}/dx = \epsilon \left\{ \tilde{X}_{pq} \delta_{pq'} + \delta_{p\dot{e}} \tilde{X}_{pq'} \right\} \underline{y}_{qq}' \tag{24.9}$$

$$\underline{y}(0) = u_{0p} u_{q'}^* \qquad ; \quad \delta_{p\dot{e}} = \text{Kronecker delta} \rightarrow \underline{I}$$

Thus the <u>power amplitudes $\underline{y}$ of the n-vector amplitude field $\mathcal{V}$ obey a first order initial</u>
<u>value problem</u>. This may be solved by a straight forward perturbation. However, the
resultant solution will be range limited due to presence of secular terms. To extend the
range of validity, the range coordinate is stretched, i.e. one replaces range coordinate x
by ξ/ϵ^2, and $\underline{y}(x)$ by $\underline{y}^{(\epsilon)}(\xi) = \underline{y}\left(\frac{\xi}{\epsilon^2}\right)$ Then, defining $\underline{w}(\xi)$ as

$$\underline{w}(\xi) = \lim_{\epsilon \to 0} \langle \underline{y}^{(\epsilon)}(\xi) \rangle \tag{24.10}$$

one arrives at a system of equations in the n-vector $\underline{w}(\xi)$ of scalar ξ (= range),

$$\frac{d\underline{w}(\xi)}{d\xi} = \underline{\underline{V}} \, \underline{w}(\xi), \qquad \underline{w}(0) = \underline{u}_0 \otimes \underline{u}_0^*$$

$$\underline{\underline{V}} = \lim_{x \to \infty} \frac{1}{x} \int_0^x \int_0^s \langle \underline{\underline{V}}(s) \underline{\underline{V}}(\sigma) \rangle \, d\sigma \, ds \tag{24.11}$$

$$\underline{w}(\xi) = (w_{pp}(\xi)), \qquad W_p = w_{pp}(\xi)$$

By <u>using</u> Eq. (24.8) for the definition of $\underline{\underline{V}}(x)$, and by performing the limiting operation
for $\underline{\underline{V}}$, it is seen that the mean power amplitudes (= the diagonal parts of $\underline{w}(\xi)$)
obey a set of <u>kinetic equations</u>,

$$\frac{dW_p}{d\xi} = \sum_{q=1}^{n} Q_{pq} \, W_q, \qquad W_p(0) = |u_{0p}|^2, \quad p = 0,1,2 \cdots \tag{24.12}$$

$$Q_{pq} = \delta_{pq} \left(\int_0^\infty e^{-i(k_q - k_r)\sigma} R_{pr,nq}(\sigma) d\sigma + \int_0^\infty e^{i(k_q - k_r)} R_{pq,pq}^{*,*}(\sigma) d\sigma \right)$$

$$+ \int_0^\infty e^{i(k_q - k_p)} R_{pq,pq}^{,*}(\sigma) d\sigma + \int_0^\infty e^{-i(k_q - k_p)} R_{pq,pq}^{k,}(\sigma) d\sigma \tag{24.13}$$

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in which the correlation functions R for the (modally expressed) random index of refraction are defined as

$$R^{*,*}_{P\vec{q},P'\vec{q}'}(\sigma) = \langle X^{*}_{P\vec{q}}(\sigma+x) X^{*}_{P'\vec{q}'}(x) \rangle, \text{ etc.} \qquad (24.14)$$

Upon application of Eqs. (24.11) thru (24.14) to Eq. (24.5), it is seen that

$$W_P(\xi) = \lim_{\epsilon \to 0} \langle |v_p(\tfrac{\xi}{\epsilon^2})|^2 \rangle$$

$$dW/d\xi = \sum_{q=1}^{n} Q_{pq} W_q, \qquad Q_{pp} = -\sum_{p \neq p} Q_{pq} \qquad (24.15)$$

$$Q_{pq} = 2 \int_0^{\infty} [\cos(k_p - k_q)] S_{P\vec{q},\vec{q}p}(\sigma) d\sigma, \qquad p \neq q$$

$$S_{P\vec{q},\vec{q}p} = \langle \mu_{P\vec{q}}(\sigma+x) \mu_{\vec{q}q}(x) \rangle$$

This set of equations is the key result of the method. It gives the mean power modal amplitudes of the field in the limit $\epsilon \to 0$, $x \to \infty$ such that $\epsilon^2 x = \xi$ = const..

From Eq. (24.15) it is noted that

$$\sum_{p=1}^{n} W_p(\tfrac{b}{\xi}) = \text{const.} \qquad (24.16)$$

If the constant in Eq. (24.16) is taken as unity then $W_P(\xi)$ can be interpreted as probabilities, and this equation is a "foward Kolmogorov equation" for a continuous range Markov chain. The transition probabilities of this chain describe how power is transferred among the modes, as prescribed by Eq. (24.12). Now the physical dimensions of Q_{pp} is reciprocal length. Hence, if α is defined as the maximum value of $-Q_p^{-1}$ then α is largest mean distance between transitions of the chain. If we chose ξ to be much larger than α then the power in the p'th mode for each of n modes is $W_p = \frac{1}{n}$, i.e., the power is equally partitioned among the modes.

25. Method of Diffracting Screens I

A planar diffracting screen is defined as a parallel plane stratum, thin in the Z direction, and indefinite extent in the $X Y$ direction. A plane wave $A e^{i\phi}$ incident upon this screen is modified by the physical properties of the screen so that it leaves the screen in a scattered (or diffracted) state, $f(X,Y) = A(X,Y) \exp\{i\phi(X,Y)\}$. At the exit plane ($Z = 0$), the function $f(X,Y)$ can be expanded in a Fourier spectrum $F(\underline{K}_T)$ according to the relation,

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$$f(\underline{X}_T, \omega) = \iint_{-\infty}^{\infty} F(\underline{K}_T) \exp i \left\{ \underline{K}_T \cdot \underline{X}_T - \omega t \right\} \frac{d^2 \underline{K}_T}{(2\pi)^2} \tag{25.1}$$

$$\underline{X}_T = X\hat{\imath} + Y\hat{\jmath} \; ; \quad \underline{K}_T = K_x \hat{\imath} + K_y \hat{\jmath} \; ; \quad d^2\underline{K}_T = dK_x dk_y$$

$$K_x = |\underline{K}_T| \cos \theta_1 \; ; \quad K_y = |\underline{K}_T| \cos \theta_2 \; ; \quad |\underline{K}_T| = 2\pi/\lambda$$

Since

$$\underline{K}_T \cdot \underline{X}_T = \frac{2\pi}{\lambda} \left[X \cos \theta_1 + Y \cos \theta_2 \right] \tag{25.2}$$

it is seen that $f(\underline{X}_T)$ is a sum of plane waves oriented in two dimensional space in directions specified by the direction cosines ($\cos \theta_1$, $\cos \theta_2$). Thus $F(\underline{K}_T)$ defines the spectral properties of the wave amplitude distribution in the screen and $f(\underline{X}_T)$ describes the spatial aperture function or resultant diffracted wave as a sum of plane waves (or angular spectrum of waves) traveling in different directions. When $Z > 0$ the diffracted wave field has the form

$$f(\underline{X}_T, Z; \omega) = \iint_{-\infty}^{\infty} F(\underline{K}_T) \cdot \exp \left[\underline{K}_T \cdot \underline{X}_T + K_z Z - \omega t \right] \frac{d^2 \underline{K}_T}{(2\pi)^2} \tag{25.3}$$

Here $\underline{K}_T$ and K_z are the components of the 3-dimensional propagation constant $\underline{K}$,

$$\underline{K} = K_x \hat{\imath} + K_y \hat{\jmath} + K_z \hat{k} \; , \qquad K_z = |\underline{K}| \cos \theta_3$$

The screen function $F(\underline{K}_T)$ may be modeled as line and/or continuous spectra. For example, a cosine diffracting screen is defined by a F which consists of two line spectra,

$$F(\underline{K}_{oT})$$

$$F(\underline{K}_{oT}) = \delta(\underline{K}_T - \underline{k}_{oT}) \frac{1}{2} a\, e^{i |\underline{\phi}_{oT}|}$$

$$F(-\underline{K}_{oT}) = \delta(\underline{K}_T + \underline{K}_{oT}) \frac{1}{2} a\, e^{-i |\underline{\phi}_{oT}|} \tag{25.4}$$

$$\underline{\phi}_T = \phi_x \hat{\imath} + \phi_y \hat{\jmath}$$

The resultant disturbance $f(\underline{X}_T, Z, \omega)$ is

$$f(\underline{X}_T, Z, \omega) = a \cos \left\{ \frac{2\pi}{\lambda} \left[X \cos \theta_1 + Y \cos \theta_2 + |\underline{\phi}_T| \right] \right\}$$

$$\times \exp i \left\{ \frac{2\pi}{\lambda} Z \cos \theta_3 - \omega t \right\} \tag{25.5}$$

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If we regard the incident plane wave as normal (i.e. $\theta = 0, \beta = 0,$ (in spherical coordinates)) to $Z = 0$, then the effect of the screen is to turn the propagation vector of the incident wave through a space angle (θ, β) (referred to the Z-axis). Thus the angle of $F\left(\underline{K}_T\right)$ $d^2\underline{K}_T$ expresses how much turning is effected by the screen in the incremental wave number $d\underline{K}_T$. It is easily seen that the direction cosine $\cos\theta_2 = \sin\theta_1$. If the distance X is taken as one spatial period ($= L_x$) then in the X direction,

$$\left(\frac{2\pi}{\lambda} \sin\theta\right) L_x = 2\pi, \quad \text{or} \quad L_x = \frac{\lambda}{\sin\theta} \tag{25.6}$$

The angle θ has this significance: a group of plane waves propagating at various angles θ_i interfere to produce a spatial pattern of amplitude. In particular, such patterns describe the amplitude in the plane ($Z = 0$) of a diffracting screen. The periodicity L_x of the wave in the plane $Z = 0$ is given by (25.6). When $|\sin\theta| > 1$ it is seen that L_x is smaller than the wave length of the incident wave, and that K_z (which is proportional to $\cos\theta$) is imaginary. Thus when $|\sin\theta| > 1$ the disturbance is exponentially attenuated in the Z direction, and the spatial period of the screen in the X direction is less than a wavelength of the incident wave. This is the origin of the evanescent wave field found in the immediate vicinity of the screen. These remarks hold not only for a cosine screen but for diffracting screens in general. Evanescent fields are discussed in greater detail below.

Eq. (25.1) expresses the field across the plane $Z = 0$ as a superposition of the $Z = 0$ traces of plane waves travelling in all the directions $\underline{K}/|\underline{K}|$ at frequency ω, and amplitude $F(\underline{K}_T)$. All the waves extend simultaneously from $-\infty$ to $+\infty$ and it is their values at $Z = 0$ that add up to give the total field. The angle θ which these waves make with the Z axis must be allowed to become complex to account for near-field effects. Eq. (25.1) can also be interpreted as a scan through all the space periodicities L (where $|\underline{K}_T| = \frac{2\pi}{L}$) of the disturbing element in the diffracting screen. When $\underline{K}_T$ is small, L is large, and as $L \to \infty$ the XY component of the disturbed wave field approaches zero showing that the (normal) incident wave passes through without deflection. Thus diffracting screens having very large spatial periodicities relative to the period of the incident wave (of wavelength λ) have little effect on the passage of incident plane waves. As $L \to \lambda$ the magnitude of $\underline{K}_T \cdot$ X_T for fixed $|X_T|$ increases, showing that spatial periodicities in the screen which are larger than, but near in size to, the wave length of the incident wave turn the incident

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wave vector through an angle whose tangent is constructible from an ever increasing XY component. When $L = \lambda$ the constituent plane waves are all traveling parallel to the XY plane, i.e. the vector $\underline{K}/|\underline{K}|$ is parallel to the XY plane. When $L < \lambda$, the angle θ is complex. In this case the constituent plane waves resulting from diffraction travel parallel to the XY phase but are exponentially attenuated in the Z direction, i.e. they are (as noted in the previous paragraph) <u>evanescent</u> waves. Thus when the spatial periodicities of the diffracting screen are less than a wave length of the incident wave (alternatively when the aperture distribution is described by wave numbers larger than that of the incident wave) the disturbance due to the screen is not propagated to the far field, that is, diffracting obstacles of size less than the incident wavelength cannot be resolved. The wave disturbances due to such obstacles are found only in the near field of the diffracting screen.

If $f(\underline{X}_T, \omega)$ represents the complex screen aperture function then the screen autocorrelation function has the form

$$\varrho(\underline{d}_T) = \frac{\int_{-\infty}^{\infty} f(\underline{X}_T, \omega) \, f^*(\underline{X}_T + \underline{d}_T, \omega) \, d\underline{X}_T}{\int_{-\infty}^{\infty} f(\underline{X}_T, \omega) f^*(\underline{X}_T, \omega) \, d\underline{X}_T} \tag{25.7}$$

Now when $\varrho(\underline{d}_T)$ and the screen spectral function $F(\underline{K}_T)$ are properly normalized, then by the Wiener-Kinchin theorem $\varrho(\underline{d}_T)$ and $|F(\underline{K}_T)|^2$ are Fourier transforms (symbol $\mathcal{F}$) of each other. Thus in the plane of the diffracting screen

$$\varrho(\underline{d}_T) = \mathcal{F}\left\{ |F(\underline{K}_T)|^2 \right\} \tag{25.8}$$

$$\underline{d}_T = \xi \hat{\imath} + \eta \hat{\jmath}, \qquad \underline{d} = \underline{d}_T + \zeta \hat{k}$$

In the plane $Z = \zeta = \text{const.}$

$$\varrho(\xi, \eta, \zeta) = \iiint |F(\underline{K}_T)|^2 \exp \iota \left[\underline{K}_T \cdot \underline{d}_T + K_z \zeta - \omega t \right] \frac{d^2 \underline{K}_T}{(2\pi)^2}$$

Since, $\tag{25.9}$

$$K_z = |K| \sqrt{1 - \frac{|K_T|^2}{|K|^2}} = |K| \sqrt{1 - \alpha^2}$$

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it is seen that for small α ,

$$K_z \approx |K| \left(1 - \frac{|K_T|^2}{2|K|^2} \right) \tag{25.10}$$

Now let $\exp i K_z \zeta$ be the Fourier transform of a function $\hat{h}(x)$ i.e. define

$$\hat{h}(\underline{x}) \equiv \int\!\!\!\int_{-\infty}^{\infty} (\exp i K_z \zeta) \exp i \left[\underline{K}_T \cdot \underline{d}_T \right] \frac{d^2 \underline{K}_T}{(2\pi)^2} \tag{25.11}$$

then we see from (25.3) that

$$\mathcal{F}\{ f(\underline{X}_T, Z) \} = F(\underline{K}_T) H(\underline{K}_-, Z) \tag{25.12}$$

or

$$f(X_T, Z) = \int\!\!\!\int_{-\infty}^{\infty} f(X, Y) h(X - \xi, Y - \eta, Z) d\xi d\eta \tag{25.13}$$

In the Fresnel approximation

$$h \approx \exp i |K| \left[\frac{X^2 + Y^2}{2Z} \right] \tag{25.14}$$

Thus the Fourier transform of $f(X_T, Z)$ differs from that of $f(X_T)$ only by a complex exponential. From this it is seen that the autocorrelation function of complex $f(X_T, Z)$ for any plane Z =const. is the same as the autocorrelation function over the diffracting screen itself. Alternatively, it is seen that the amplitude spectrum $F(K_T)$ of the Fresnel field is the same for all planes Z=const. in that field, only the relative phase of the field being altered as Z is changed.

26. Method of Diffracting Screens II

A thin slab of thickness ΔZ is assumed to have a (total) index of refraction $m(x)$ whose square is given in a first approximation, by

$$m^2(x) \approx m_o^2 + \mu(x) \qquad \mu(x) = |\mu(x)| e^{i\psi} \tag{26.1}$$

By differentiation it is seem that the (finite size) departure from the mean value of m is

$$\Delta m(x) = \frac{1}{2m} \Delta \mu(x) \tag{26.2}$$

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A plane wave traveling in the Z-direction enters the slab at normal incidence with unit amplitude and zero phase, traverses it, and exits with a complex amplitude $f(x)$ given by

$$f(x) = exp \left\{ i \, \Delta \varphi (x) \right\} \tag{26.3}$$

The phase change $\Delta \varphi(x)$ of the plane wave is

$$\Delta \varphi(x) = k \, \Delta Z = \left[\frac{\pi}{n \lambda} \Delta \mu (x) \right] \Delta Z \tag{26.4}$$

For small absorption by the medium the index n is real and approximately equal to n_0, and the wave length approximately equal to λ_0, so that

$$\Delta \varphi(x) \approx C \, \Delta \mu (x) , \qquad C = \frac{\pi}{n_0 \lambda_0} \Delta Z \tag{26.5}$$

Now let $\Delta \mu(x)$ be a random function and assume it is normally distributed with a standard deviation $(\Delta \mu)_m$, and an autocorrelation $R_\mu(\xi)$. According to (26.5) it is seen that $|\Delta \varphi(x)|$ is also a random function. Let its standard deviation be ϕ_m. An explicit relation between the autocorrelation of $f(x)$ and that of $\Delta \mu(x)$ has been derived by Bramley (1955). It is

$$R_f (\xi) = exp \left[- \phi_m^2 \left\{ 1 - R_\mu (\xi) \right\} \right] \tag{26.6}$$

In these formulas we define $R_f(\xi)$ by the formula

$$R_f (\xi) = \frac{\langle f(x) f^*(x+\xi) \rangle}{\langle f(x) f^*(x) \rangle} \tag{26.7}$$

If $f(x)$ is the sum of a stationary mean ($= a$) and a random function with zero mean ($= g(x)$), then by (26.7) it is seen that

$$R_f (\xi) = \frac{b + R_g (\xi)}{b+1} \tag{26.8}$$

where

$$b = |a|^2 \Big/ | \langle g(x) g^*(x) \rangle | \tag{26.9}$$

and $R_g(\xi)$ is the autocorrelation of $g(x)$. Eqs. (26.8) and (26.9) permit comparisons with (26.6) for two important cases. In the first case let $\phi_m^2 \ll 1$. Then

$$R_f (\xi) = 1 - \phi_m^2 \left(1 - R_\mu (\xi) \right) \tag{26.10}$$

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From this it is seen that

$$f = \frac{1}{\phi_m^2} \quad ; \quad R_g(\xi) = R_\mu(\xi) \tag{26.11}$$

Since R_g and R_μ are identical one infers (when $\phi_m^2 \ll 1$) that the scale of the random part of $f(x)$ is the same as the scale of the random part in the index of refraction. In the second case let $\phi_m^2 \gg 1$, and assume a scale size ξ_0 such that

$$R_\mu(\xi) = e^{-\xi^2/\xi_0^2} \tag{26.12}$$

If $\xi^2 \gtrsim \xi_0^2$ then $R_\mu(\xi) \to 0$ and $R_f(\xi)$ is constant $(= e^{-\phi_m^2})$ and small. The important values of R_f are thus related to the requirement that $\xi^2 \ll \xi_0^2$. In the latter case

$$1 - R_\mu \approx \xi^2/\xi_0^2 \tag{26.13}$$

so that

$$R_f(\xi) = e^{-\phi_m^2 \xi^2/\xi_0^2} \tag{26.14}$$

Thus the scale size of the random function $f(x)$ is equal to ϕ_m^{-1} times the scale size ξ_0 of the random index of refraction.

We next return to Eq. (26.6) and assume R_μ to be given by (26.9). The power spectrum of $f(x)$ (namely $|F(k_T)|^2$) is found by taking the Fourier transform of the autocorrelation function (= Eq. (26.6)).

For propagation wave vector $\underline{K}$, and azimuthal angle β, and polar angle θ the component wave vector $\underline{K}_T$ is given by

$$\underline{K}_T = |K| \left\{ \sin\theta \cos\beta \, \hat{\imath} + \sin\theta \sin\beta \, \hat{\jmath} \right\} \tag{26.15}$$

Assuming (for convenience) only x-dependence (i.e. $\beta = 0^\circ$) we write the power spectrum of $f(x)$ as $|F(S)|^2$ where $S = \sin\theta$. For the assumed autocorrelation function

$$R_f(\xi) = e^{-\phi_m^2 [1 - \xi^2/\xi_0^2]}$$

Bramley (1955) has shown that the angular power spectrum is given by

$$|F(s)|^2 = e^{-\phi_m^2} \left[\delta(S) + \pi^{\frac{1}{2}} \xi_0 \sum_{n=1}^{\infty} \frac{\phi_m^{2n}}{n!} \frac{1}{n^{\frac{1}{2}}} \exp\left(-\frac{\pi \xi_0^2 S^2}{n}\right) \right] \tag{26.16}$$

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The interpretation of this formula lends insight into the physical process of the scattering by random inhomogeneities. The first noticeable feature is the delta function $\delta(S)$ which upon integration over θ has a value of unity when $S=0$ i.e. it represents the undeviated wave. The remaining terms assembled inside the summation sign represent a continuous power spectrum of "noise" created by scattering of obstacles composing the (random) screen. If $\phi_m^2 \ll 1$ then only the $m=1$ term is needed to compute the noise. The power spectrum in that case is

$$|F(s)|^2 = e^{-\phi_m^2}\, \delta(S) + \pi^{\frac{1}{2}} \xi_o \phi_m^2\, e^{-\phi_m^2} exp\left(-\pi \xi_o^2 S^2\right) \quad (26.17)$$

i.e. the noise is Gaussian distribution, of the same form as the probability distribution of the random index of refraction. The signal to noise power ratio is

$$b = e^{-\phi_m^2}\Big/\left(1 - e^{-\phi_m^2}\right) \quad (26.18)$$

The power spectrum derived in Eq. (26.16) for a single thin diffracting screen is the principle result of the analysis of this section. It will be used to derive further results for the case of thick screens.

27. Phase Diffracting Screen III (Bramley, Proc. Roy. Soc. $\underline{225}$ 1954 (515)).

A plane wave $E(x,t)$ of known fixed amplitude and zerophase is normally incident on a plane diffracting screen which imposes a random phase $\phi(x)$ variation on the wave as a function of position. Let the random phase $\phi(x)$ be normally distributed over the screen, i.e. let its probability density be given by

$$f(\phi) = \frac{1}{b\sqrt{2\pi}}\, e^{-(\phi-\phi_0)/2\,b^2}\,, \qquad \phi = \phi(x) \quad (27.1)$$

in which ϕ_0 is the mean value of the r.v. ϕ at a point x and b is its variance at the same point. Assume next that ϕ is homogeneous and stationary. Then the autocorrelation $R_\phi(\lambda)$ of ϕ between two points x, x_2, separated by a distance λ is related to the square of the phase difference at these points by the formula [Papoulis, 1965 p. 337],

$$\chi_o^2(\lambda) \equiv \left\langle \left[\phi(x+\lambda) - \phi(x)\right]^2\right\rangle = 2\left[R_\phi(0) - R_\phi(\lambda)\right]$$

$$(27.2)$$

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$$'' = 2\phi_0^2 \left[\varsigma_\phi(0) - \varsigma_\phi(r) \right] \tag{27.3}$$

where ς_ϕ is the normalized correlation function. Now the plane wave exiting the screen is a complex r.v., $E(x)$. It can be shown to have a (normalized) autocorrelation,

$$\varsigma_E(r) = exp\left[-\tfrac{1}{2} \chi_0^2(r) \right] = exp\left[-\phi_0^2(1 - \varsigma_\phi(r)) \right] \tag{27.4}$$

[Hewish*]. Assume that the form of ς_ϕ is $exp\ (-r^2/\ell^2)$, i.e. assume there is a separation ℓ (= scale size) such that when $|r| \ll \ell$ the random phase ϕ is well correlated, while the separation $r \gg \ell$ means that the random phase is poorly correlated. Then Eq. (27.4) takes on the form

$$\varsigma_E(r) = exp(-\phi_0^2) \sum_{n=0}^{\infty} \frac{\phi_0^{2m}}{n!} exp\left(-n r^2/\ell^2 \right) \tag{27.5}$$

Now the (normalized) angular power spectrum of the wave E is $|F_E(S)|^2$ where $S = \sin\vartheta$ and ϑ is the angle between the incident and scattered wave. It can be calculated as the Fourier-Bessel transform of $\varsigma_E(r)$

$$\frac{|F_E(S)|^2}{\int_0^\infty |F_E(p)|^2 p\, dp} = k^2 \int_0^\infty \varsigma_E(r)\, r\, J_0(k r S)\, dr \tag{27.6}$$

Substituting (27.4) into (27.6), and then integrating, leads to

$$\frac{|F_E(S)|^2}{\int_0^\infty |F_E(p)|^2 p\, dp} = \frac{(k\ell)^2}{2} e^{-\phi_0^2} \sum_{n=0}^{\infty} \frac{\phi_0^{2m}}{n!\, n} exp\left(-\frac{\pi^2 \ell^2 S^2}{n \lambda^2} \right) \tag{27.7}$$

in which λ is the wavelength of the incident wave. This result applies to a thin screen. When transmission is through a thick irregular medium the value of ϕ is then a function of the thickness ΔZ of traversal. Assume the index of refraction consists of a sum of average and random components, i.e.

$$n^2 = n_0^2 + \mu(x) \tag{27.8}$$

Then, approximately,

$$\Delta n = \Delta \mu / 2n \tag{27.9}$$

*Proc. Roy. Soc. A 209, 81 (1951)

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so that the phase change over distance dz is

$$d\phi = k \Delta n \, dz = \frac{\pi \Delta \mu}{\lambda n} dz \qquad (27.10)$$

Since $\dfrac{d\phi}{dz}$ is a r.v., we take its variance to be $(\pi^2/\lambda^2) \langle \Delta\mu \Delta\mu^* \rangle / n^2$. The autocorrelation $\mathcal{C}_\phi(\eta)$ of ϕ is taken (as before) to be $\exp(-\hbar^2/\ell^2)$. Now the variance of $d\phi/dz$ is taken at a specific point X while the correlation of ϕ is taken for a separation $\hbar$ (in the X-direction). We desire to find the variance of ϕ over the propagation interval ΔZ by integrating over this interval. To do this we replace $\hbar$ by $iu\Delta Z$ in the formula for $\mathcal{C}_\phi$, multiply $\mathcal{C}_\phi(iu\Delta Z)$ by $1-u$, integrate over u, in the range 0 to 1, then multiply the result by twice the variance of $\dfrac{d\phi}{dz}$. In symbols*

$$\phi_0^2 = \langle \phi(x) \phi^*(x) \rangle = 2 \frac{\pi^2}{\lambda^2 n^2} \langle \Delta\mu \Delta\mu^* \rangle \int_0^1 (1-u) \exp\left(\frac{u^2 \Delta Z}{\ell^2}\right) du \qquad (27.11)$$

$$= \frac{\pi^2}{\lambda^2} \frac{\langle \Delta\mu \Delta\mu^* \rangle}{n^2} \left[\pi^{1/2} \ell \, \Delta Z \, erf \frac{\Delta Z}{\ell} - \ell^2 \left(1 - \exp\frac{(\Delta Z)^2}{\ell^2}\right) \right] \qquad (27.12)$$

When $\Delta Z \gg \ell$, i.e. when the range is much greater than the scale of irregularities of the index of refraction, then

$$\phi_0^2 = \pi^{5/2} \ell \langle \Delta\mu \Delta\mu^* \rangle \Delta Z / \lambda^2 n^2 \qquad (27.13)$$

This is the variance of ϕ averaged over a distance ΔZ, taken at a specific point in X. It is seen that in this model the variance of ϕ increases linearly with distance. According to Eq. (27.5) the autocorrelation of the wave is proportional to $\exp(-\phi_0^2)$. Thus $\mathcal{C}_E(0)$ diminishes indefinitely with distance. In contrast, the autocorrelation $\mathcal{C}_\phi(\eta)$ can be shown to retain its form ($= \exp(-\frac{\hbar^2}{\ell^2})$) regardless of distance ΔZ in the direction of propagation.

28. Fokker-Planck Equation

Let $X(t)$ be a random variable of the continuous type, and let its conditional probability density be $p(x,t \mid x_0, t_0)$. Assuming that $\underline{x}(t_0) = x_0 \hat{\imath}$ one can define the conditional mean, and conditional variance of $x(t)$ to be $a_x(x_0, t, t_0)$, $b_x(x_0, t, t_0)$ respectively. The slopes of a and b are given special notations,

*[Bramley Proc. Roy. Soc. A 225, 515 (1954)]

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$$\eta (x,t)= \partial a/\partial t \quad ; \quad \sigma^2 = \partial b/\partial t \tag{28.1}$$

We now restrict $\underline{x}(t)$ to be a Markov process, that is, we restrict the joint probability distribution $\mathcal{P}\{\underline{x}(t_m) \leq x_n |\underline{x}(t_{m-1})\}$ to be the <u>single</u> member $\mathcal{P}\{\underline{x}(t_m) \leq \underline{x}_n(t_m) |\underline{x}(t_{n-1})\}$. The Markov process $\underline{x}(t)$ is associated with a first order differential equation,

$$\frac{d\underline{x}}{dt} - \underline{\beta}(\underline{x},t) = \underline{g}(t) \tag{28.2}$$

in which the forcing functions $\underline{g}(t_k)$ for all $t_1, t_2 \cdots t_m$ are statistically independent. When $\underline{\beta}(\underline{x},t) = -\underline{\dot{\beta}}(\underline{x},t)$ Eq. (28.2) becomes the Langevin equation. The product $\underline{g}(t)dt = w(t)$ (i.e., white noise). The conditional probability density of a process which satisfies Eq. (28.2) is known to satisfy the Fokker-Planck equation,

$$\frac{\partial \mathcal{P}}{\partial t} = \frac{1}{2} \frac{\partial^2}{\partial x^2}\left[\sigma^2(x,t)\mathcal{P}\right] - \frac{\partial}{\partial x}\left[\eta(x,t)\mathcal{P}\right] \tag{28.3}$$

(Papoulis 1965, p. 538). This equation plays a significant role in several different approaches to the solution of the random wave equation.

Moment Equations of the Parabolic Wave Equation with a random forcing

function [Lee, J. Math. Phy. 1974 (1431)]

In the parabolic or quasioptical approximation one can reduce the scalar wave equation in the field $u(\underline{x}, k)$ to a first order equation describing narrow angle forward scattering in the propagation direction z,

$$\frac{\partial}{\partial z} u(\underline{x},k) = -\frac{1}{2ik}\nabla_\perp^2 u(\underline{x},k) - \frac{1}{2ik}\mu(\underline{x})u(\underline{x},k)$$

$$\underline{x} = (z, \varsigma) \tag{28.4}$$

Let $\mu(\underline{x})$ be delta correlated in the z-direction, i.e. let the correlation scale of $\mu(x)$ in the z-direction be much smaller than the correlation scale of u in the z-direction. Under this assumption Eq. (28.1) is a first order d.e. in z for the random process $u(\underline{x}, k)$ driven by an equivalent white noise process (See Sect. 12). Hence the probability density, or distribution, of u satisfies a Fokker-Planck equation. The function $u(\underline{x}, k)$ however has three variables, z, ς, k, while the corresponding Eq. (12.9) of Sect. 12 has only one variable ($= t$). To eliminate ς, k explicitly from consideration one may introduce the

idea of a <u>characteristic functional</u> Ψ of the random field $u(z,\varsigma,k)$[See Hopf, J. Rat. Mech. Anal. 1, 87 (1952)]. This is done as follows: let $\nu(\varsigma,k)$ be an <u>arbitrary</u> function of ς, k and ν^* its complex conjugate. Next a function is formed,

$$D_z(z,\nu,\nu^*) = \iint\limits_{-\infty}^{\infty} \left[u(z,\varsigma,k)\nu(\varsigma,k) + u^*(z,\varsigma,k)\nu^*(\varsigma,k) \right] d^2\varsigma\, dk$$

(28.5)

Since u is random, it follows that D_z is random. Then a functional is formed from the r.v. D_z by writing

$$\Psi(z,\nu,\nu^*) = \langle \exp i D_z \rangle$$

(28.6)

The objective now is to derive a Fokker-Planck equation for Ψ. To do this one differentiates Eq. (28.3) with respect to z, and then uses Eq. (28.1) to replace du/dz in all subsequent development. By extended use of the theory of functional derivatives (symbol $\delta/\delta z, \frac{\delta}{\delta \nu}$) Lee (loc. cit.) finally arrives at the following F-P equation for the quantity Ψ,

$$\frac{\delta \Psi}{\delta z}(z,\nu,\nu^*) = \frac{i}{2} \int \frac{ds}{k} \left[\nu(s) \nabla_\perp^2 \frac{\delta \Psi}{\delta \nu} - \nu^*(s) \nabla_\perp^2 \frac{\delta \Psi}{\delta \nu^*} \right]$$
$$- \frac{1}{4} \int \frac{ds\, ds'}{k\, k'} A(\varsigma-\varsigma') \hat{M}(s) \hat{M}(s') \Psi$$

(28.7)

in which

$$s = (\varsigma, k)$$

$$A(\varsigma-\varsigma') = \int_{-\infty}^{z} \langle \mu(z,\varsigma)\mu(z',\varsigma)\rangle dz'$$

$$\hat{M}(s) = \nu(s)\frac{\partial}{\partial\nu(s)} - \nu^*(s)\frac{\delta}{\delta\nu^*(s)}$$

Now in view of the definition of D_z (See Eq. (28.2)), we may expand (28.3) as a double power series,

$$\Psi(z,\nu,\nu^*) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{i^{m+n}}{m!\,n!} \left(\int u(z,s)\nu(z,s)ds \right)^m \left(\int u^*(z,s')\nu^*(z,s')ds' \right)^n$$
$$= \sum\sum \frac{i^{m+n}}{m!\,n!} \left[\int\cdots\int \Gamma_{m,n}(z,s_1,s_2,\cdots s_m; s_1',s_2'\cdots s_n') \right.$$
$$\left. \times \nu_1 \cdots \nu_m \nu_1^* \cdots \nu_n^* ds_1 \cdots ds_m ds_1' \cdots ds_n' \right.$$

(28.8)

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where the symbol $\Gamma_{m,n}$ represents the m, n'th moment of the random field $u\,(z,s)$ i.e.,

$$\Gamma_{m,n} = \langle\, u_1,\, u_2,\, \cdots\, u_m\, u_1^*\, u_2^* \cdots \cdots u_m^* \rangle \tag{28.9}$$

When (28.8) and (28.9) are substituted into (28.7) the result is a zero value double sum over a series of m, n'th order integrals in $d\nu^m d\nu^n$. Since the $\nu's$ are arbitrarily defined, each m, n'th integrand is also zero. This leads to the following differential equation for the moment function

$$\frac{\partial \Gamma_{m,n}}{\partial z}(\,z,\,s_1\,\cdots s_m\, s_1'\,\cdots s_m'\,) = \frac{i}{2}\left[\,\frac{\nabla_1^2}{k_1} + \cdots + \frac{\nabla_m^2}{k_m} - \frac{\nabla_1'^2}{k_1'} - \cdots \frac{\nabla_m'^2}{k_m}\,\right]\Gamma_{m,n}$$

$$-\frac{1}{4}\left(\sum_{i=1}^{m}\sum_{\ell=1}^{m}\frac{A(\varrho_i-\varrho_\ell)}{R_i R_\ell} - \sum_{i=1}^{m}\sum_{\ell=1}^{n}\frac{A(\varrho_i-\varrho_\ell')+A(\varrho_\ell'-\varrho_i)}{k_i k_\ell}\right.$$

$$+\left.\sum_{i=1}^{n}\sum_{\ell=1}^{n}\frac{A(\varrho_i-\varrho_\ell')}{k_i R_j}\,\right)\Gamma_{m,n} \tag{28.10}$$

Eq. (28.10) is a complete set of moment equations of the random field $\mathcal{U}(z,\varrho,R)$ featuring different transverse coordinates s, and different wave numbers k.

29. Model of Middleton I (4th Int. Symp. on Multivariate Analyses, 1975).

The behaviour of the sound pressure at a point in a medium having random inhomogeneities in both volume and bounding surfaces depends not only on the pressure at the point and neighboring points, but also on its value at every other point generating coincident arrivals in retarded time. Placing dependence on point and neighboring points on the l.h.s. and values at every other point on the r.h.s. Middleton writes the scattered pressure field p_{scat} at point $\underline{R}$ and time t in the form of a differential-Stieltjes integral equation,

$$\left(\nabla^2 - \frac{1}{C_0^2}\frac{\partial^2}{\partial t^2}\right)p_{scat}\,(\underline{R},t) = \frac{A_m}{C_0^2}\int_\tau d\tau \int_Z dN(\underline{Z},\tau)\,\hat{k}_m(t-\tau|Z)\frac{d^2 p_T}{dt^2}(\underline{Z},\tau) \tag{29.1}$$

Thus the scattering problem is modeled as a radiation problem due to a space-time collection of fictitious simple sources (r.h.s.). Here A_m is a constant related to the source, C_0 the speed of propagation of scalar wave p, dN is a linear sum of hierarchy of independent random point processes $dN^{(k)}$ describing the inhomogeneities of the medium, $\underline{Z}$ is a set of spatial vectors, describing location and shape of source (volume (and/or interface) scatterers), and receiver, $\hat{k}_m$ is a filter function with "memory" τ, time variation t, and spatial dependence $\underline{Z}$, and p_T is the total field pressure at the scatterer. For linear

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systems p_T is a sum of incident and scattered pressure,

$$p_T = p_{inc} + p_{scat} \tag{29.2}$$

In the symbol $dN^{(k)}$ for the random medium–descriptors the superscript k describes the order of scatter. Thus $dN^{(1)}$ describes a collection of random point scatteres in a domain Λ_M each point independent of all other points, the radiation from which obey Poisson statistics. $dN^{(2)}$ describes a collection of independent paired (i.e. correlated) point scatterers in the <u>same</u> domain Λ_M. Similarly $dN^{(3)}$ is a collection of triples the radiation of which is correlated among its three elements, but is completely independent of all other triples in the <u>same</u> domain (Λ_M). The symbol $dN^{(0)}$ represents the coherent (or specular) component of radiation.

In the linear case, Eqs. (29.1) and (29.2) can be combined and succinctly written as a Langevin-type equation,

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right) p_{scat} = \hat{g} * \left[\ddot{p}_{inc} + \ddot{p}_{scat}\right] / c_0^2 \tag{29.3}$$

in which $\hat{g}$ is the double integral operator of Eq. (29.1), and *signifies temporal convolution. The symbol $\hat{g}$ changes its form when the scattering process switches from volume to surface. Four special cases of interest to underwater acoustics can be specified:

I. <u>Single scatter at an interface.</u>

The appropriate equation is

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right) p_{scat}(\underline{R},t) = \hat{g}_{interface} * \frac{\ddot{p}_{inc}}{c_0^2} \tag{29.4a}$$

II. <u>Single scatter in a volume</u> (1st Born approximation)

This case is written as,

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right) p_{scat}(\underline{R},t) = \hat{g}_{vol} * \ddot{p}_{inc} / c_0^2 \tag{29.4b}$$

III. <u>Multiple scatter in volume.</u>

The equation of propagation is

$$\left(\nabla^2 - \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right) p_{scat}(\underline{R},t) = \hat{g}_{vol} * \left[\frac{\ddot{p}_{inc} + \ddot{p}_{scat}}{c_0^2}\right] \tag{29.4c}$$

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IV. (a) Single scatter from the interface followed by multiple scatter from the volume.

$$\left(\nabla^2 - \frac{1}{C_0^2}\frac{\partial^2}{\partial t^2}\right)\hat{p}_{scat_{vol}} = \hat{\partial}_{vol} * \left[\frac{\overset{''}{\hat{p}}_{scat-interace} + \overset{''}{\hat{p}}_{scat-vol}}{C_0^2}\right] \qquad (29.4d)$$

(b) Scatter from volume followed by scatter from surface.

The model is,

$$\left(\nabla^2 - \frac{1}{C_0^2}\frac{\partial^2}{\partial t^2}\right)\hat{p}_{scat-interace} = \hat{g}_{interace} * \frac{\overset{''}{\hat{p}}_{scat-vol}}{C_0^2} \qquad (29.4e)$$

Solutions of Special Cases

In cases I and II, the solution is directly obtained by use of generalized Fourier transforms, Green's functions and inverse operators, assuming that the hierarchy $dN^{(k)}, k = 1, 2, \ldots,$ is known. Note that $dN^{(0)}$ is omitted (i.e. the <u>specular</u> term is not treated). When ϕ scat is found (either for surface scattering or volume scattering) the received waveform $X(t)$ is obtainable by spacetime convolution, i.e.

$$X(t) = \hat{h}_R * \hat{p}_{scat} = \int_{V_R} d\eta \int_{-\infty}^{\infty} \hat{h}_R(\eta, t-\tau)\,\hat{p}_{scat}(r(\eta), \tau)\,d\tau \qquad (29.5)$$

in which $\hat{h}_R$ is the space-time operator representing the action of the receiving <u>aperature</u> on the signal ϕ scat.

An important feature of Eq. (29.5) is that since $X(t)$ and ϕ scat are simple linear sums of <u>independent</u> components (because $dN^{(k)}$ is a sum of independent $dN^{(j)}$) the moments of $X(t)$ and ϕ scat are linear functionals of moments of $dN^{(k)}$. The degree ($= m$) of the moment which is to be determined contains no higher order components of $dN^{(k)}$ than $k = m$. This is due to the assumption of single scattering i.e. no further interaction between scattered field and scatters.

The solutions of Cases III and IV for the scattered pressure are complicated by multiple scattering. Using the method of inverse operators, and expanding the results in a perturbation-theoretical series, Middleton derives an equation of an infinite series in the generalized Fourier transforms $\hat{p}, \hat{M}, \hat{g}, \hat{p}_{mc}$, i.e.

$$\hat{p}_{scat} = \sum_{q=0}^{\infty}\left[\hat{M}\left(\hat{g} * \frac{s^2}{C^2}\right)\right]^q \hat{M}\left(\hat{g} * \frac{s^2}{C_0^2}\hat{p}_{mc}\right) \qquad (29.6)$$

$$\hat{M} = \int_{VoL} G(R, \xi|s)\{\quad\}d\xi$$

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in which $\hat{M}$ is the Green's function operator for volume (or surface) scattering, and $*$ is the convolution in the complex s-domain, where s is the Laplace transform variable. This series is very similar to that of Tatarski and Gertsenshtein (1963).

The operator $\hat{g}$ appears to all orders $(g \geq 0)$. Hence $\not{p}$ scat (namely the inverse Fourier transform of $\overset{\wedge}{\not{p}}$ scat) is no longer a linear functional of $d\mathcal{N}$. All orders of moments of the $d\mathcal{N}$ now appear in even the lowest order moment of $\not{p}$scat. It is no longer possible to resolve the individual contributing orders of inhomogeneity $(R \geq 1)$ in the observed data.

30. <u>Model of Middleton II</u> - Acoustic Scattering from the random

moving Ocean surface.

The model in this case is represented by the Equation (29.4a), viz.

$$\left(\nabla^2 - \frac{1}{C_o^2} \frac{\partial^2}{\partial t^2} \right) \not{p}_{\substack{scat \\ (surface)}}(R, t) = g_{surface} * \frac{\overset{\prime\prime}{\not{p}}_{inc}}{C_o^2} \tag{30.1}$$

The received wave form is

$$X(t)_{surface} = \hat{h}_R * \not{p}_{\substack{scat \\ (surface)}} \tag{30.2}$$

where

$$\not{p}_{\substack{scat \\ (surface)}} = \mathcal{F}_s^{-1} \left\{ M\hat{g} * \frac{s^2 \overset{\wedge}{\not{p}}_{inc}}{C_o^2} \right\}, \qquad \mathcal{F}_s^{-1} = \begin{array}{l} \text{inverse Laplace} \\ \text{transform} \end{array}$$

To explore this model let the randomly elevated surface of insonification be domain Λ_s centered at origin O_s and let each point in this domain be located by vector $\underline{\lambda}$ as measured from the source origin O_T in the medium. At point $P(\underline{\lambda})$ in $\Lambda(s)$ let $\zeta(\underline{\lambda}, t)$ be the elevation of the surface above the mean with $\underline{\lambda}$ being measured from O_s in the plane of the mean surface. Next select two points on the random surface λ_s labelling them 1, 2 respectively. Let the total travel time from source origin O_T to surface point $P(\lambda)$ to receiver origin O_R be designated T_o. Now in accordance with (30.1) above we require a formula for $\not{p}_{inc}$. Choose a narrow band source signal $S = S_o e^{i\omega_o t}$, where $S_o = A_o U_o$ is the complex signal amplitude. The goal of the analysis is to derive an expression for the statistics of the signal waveform $X(t)$ at the receiver. An important statistical quantity is

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the covariance K of the received incoherent surface scatter X. Since the received signal $X(t|\lambda)$ from each scatter point is a random function of time we write the argument as $K(t_1, t_2)$. The fundamental physics of surface scatter involves the coupling of the incident wave P'_{m_c} with the randomly elevated surface. Thus we expect K to be a function of the product of source signal amplitude and some function of surface elevation and vertical velocity. For each elementary surface thus coupled to the incident wave there is associated a "filter" (i.e. scattering) function, namely the $f_{m} dN$ of Eq. (30.1), which essentially describes the scattering cross section and the order of point scatter which radiates to the receiver. The statistical behaviour of the surface elevation and the statistical behaviour of the scattering function (namely, its order of point scatterers) are considered <u>statistically independent</u>. Thus the covariance K of the received signal is modelled to depend on the product of the covariance C_ζ of the surface elevation and the covariance C_{dN} of the scattering function.

With these special assumptions Middleton's model of the covariance of the received signal takes on the form,

$$K(t_1, t_2) = \frac{A_o^2}{2} Re \left\{ e^{i\omega_o(t_2 - t_1)} \int_{\Lambda_s} u_o^* \left[t_1 - T_o(\underline{\lambda}_1) \right] u_o \left[t_2 - T_o(\underline{\lambda}_2) \right] \right.$$

$$\times d(\underline{\lambda}_1, \underline{\lambda}_2 | \omega_o) C_\zeta \left[\zeta(\underline{\lambda}_1), \zeta(\underline{\lambda}_2), \zeta^o(\underline{\lambda}_1), \zeta^o(\underline{\lambda}_2) | b \right] \tag{30.3}$$

$$\times C_{dN} \left[\gamma_o(\underline{\lambda}_1), \gamma'(\underline{\lambda}_2) | dN(\underline{Z}(\underline{\lambda}_1)) dN(\underline{Z}(\underline{\lambda}_2)) \right]$$

$$\times e^{i\omega \left[T_o(\underline{\lambda}_1) - T_o(\underline{\lambda}_2) \right]} d\underline{\lambda}_1 d\underline{\lambda}_2.$$

Here b is a function of "ray" angles of source and receiver, and d is a <u>geometric factor</u> accounting for "inverse-R" spreading and beam patterns of source and receiver. Explicit forms for b, d, C, γ_o, etc. are found in Middleton loc. cita.

An important feature of Eq. (30.3) is the covariance of the scattering function C_{dN}. The model selected by Middleton leads to the specific form,

$$C_{dN} = \langle \gamma_o(\underline{\lambda}_1) \gamma'(\underline{\lambda}_2) \rangle \langle dN(\underline{Z}(\lambda_1)) dN(\underline{Z}(\lambda_2)) \rangle$$

$$\tag{30.4}$$

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in which $\gamma_0(\lambda)$ is the "scattering cross-section" of the elementary areas in Λ_ζ. Expanding dN as a sum of Poisson distributed first order point scatterers, and second order coupled pairs, one arrives at the new representation,

$$C_{dN} = C_{dN}^{(1)} \delta(\lambda_2 - \lambda_1) + C_{dN}^{(2)}(\lambda_1, \lambda_2) \tag{30.5}$$

The scattering due to $C_{dN}^{(1)}$ is a constant (independent of frequency) and is dominant at high frequencies $(f_0 > 5\,kHz)$ while the scattering due to $C_{dN}^{(2)}$ is dependent on f_0^{-2} and is dominant at low frequencies $(f_0 < 1\,kHz)$.

31. <u>Phenomenological Modeling</u> [D. A. DeWolf, Proc. IEEE <u>62</u>, 1523 (1974)]

Inhomogeneities which interact with acoustic waves in fluids can be modeled as moving lens-like spheres (phenomenological modeling). Assuming the motion random we select a point $\underline{x}$ in the fluid and measure the index of refraction $n(\underline{x}, t)$ to be

$$n(\underline{x}, t) = n_0(\underline{x}, t) + \delta n(\underline{x}, t) \tag{31.1}$$

in which n_0 is a deterministic component and δn is the random component. Assume further that the random component is a sum of these (interpenetrating) lenses and thus changes its value randomly from moment to moment. If $\underline{X}_m(t)$ is the spatial location of the m'th lens at time t it will be convenient to model the spatial dependence of δn inside a specific lens as bell-shaped (i.e., gaussian shape) whose effective width will be the diameter ℓ of the sphere. For each size ℓ of the lens there will be a specific $\delta n(\ell)$. Since size is random the quantity $\delta n(\ell)$ is random. Collecting all assumptions it is seen that for each specific size ℓ, the random component of the index at $\underline{x}, t$ is,

$$\delta n_\ell(\underline{x}, t) = \sum_m \delta n(\ell) \exp\left\{ -\frac{[\underline{x} - \underline{X}_m(t)]^2}{\ell^2} \right\} \tag{31.2}$$

Let the mean value of $\delta n(\ell)$ be zero, and let $\Phi(K)$ be the power spectrum of the random function $\delta n(\ell)$ of the random variable ℓ (i.e., $K\ell$ is the Fourier transform pair). The contribution to the variance $\langle \delta n(\ell)^2 \rangle$ in K-space due to volume element $4\pi K^2 dK = dV(K)$ is $\Phi(K)\,dV(K)$. The power spectrum $\Phi(K)$ represents some physical process in nature. In many processes it it convenient to adopt a <u>Kolmogoroff spectrum</u>,

$$\Phi(K) \sim C_m^2 K^{-11/3} \tag{31.3}$$

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Assuming that for the size parameter ℓ the corresponding important K is such that $K\ell \sim 1$, one deduces that

$$\Phi(K)\,dV(K) \sim C_m^2\,\ell^{-1/3}\,d\ell \tag{31.4}$$

Thus for each size ℓ the variance of $\delta m(\ell)$ is given by,

$$\langle \delta m(\ell)^2\rangle \sim C_m^2\,\ell^{-1/3} \tag{31.5}$$

where the dimensions of C_m^2 are (length)$^{-2/3}$, and those of $\langle \delta m(\ell)^2\rangle$ are (length)$^{-1}$.

The size ℓ is a parameter which serves to introduce a scale into the modeling of the scattering process. A second scale parameter is the quantity $\ell_F = (\lambda L)^{\frac{1}{2}}$ in which λ is the wavelength of the (single frequency) acoustic wave and L the propagation distance. This quantity ℓ_F is (approximately) the radius of the first Fresnel zone on a spherical wave front radiating from the point $(0, z)$, and measured from the (reference) point $(0, L)$. The parameter ℓ_F is assumed to have an upper limit L_o equal to the macroscale of turbulence (or largest inhomogeneity), and a lower limit equal to the microscale ℓ_ρ (or smallest inhomogeneity). Collecting all scales one defines six parameters of size, $\ell, \ell_F, \ell_\rho, L_o, L, \lambda$ which play a role in the theory of propagation of sound in a turbulent medium.

Using two or more of these six size parameters one can construct auxiliary angle and size parameters. The first of these is the angle θ_F subtended by ℓ_F over the distance L, i.e. $\theta_F \sim (\lambda/L)^{1/2}$. By definition two rays separated by length ℓ_F traveling at angle θ_F to each other, interfere coherently. The second of these parameters is the angle θ_d through which a wave of wavelength λ is diffracted by a sphere of size ℓ. By construction $\theta_d \sim \frac{\lambda}{\ell}$. Since amplitude changes are proportional to angle θ the acoustic power changes are proportional to angle squared. Thus the distance $L = L_d$ beyond which diffraction power changes exceed interference power changes is given by $\theta_d^2 > \theta_F^2$, or $L_d > \ell^2/\lambda$. A third angle of importance is the angle of refraction θ_λ. For each single sphere of inhomogeneity the angle of refraction is taken to be proportional to δm. Since there are L/ℓ spheres in the length L the total angle of refraction in length L is $\theta_\lambda \sim \delta m\, L/\ell$. This is a random quantity. The acoustic power refracted $(\sim \langle \theta_\lambda^2\rangle)$ exceeds the power that undergoes interference $(\sim \theta_F^2)$ at distances greater than $L = L_\lambda$ where L_λ is determined by the condition $\langle \theta_\lambda^2\rangle > \theta_F^2$ i.e. at

$$L_\lambda \sim (\lambda \ell)^{1/2}/\delta m \tag{31.6}$$

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To summarize: the reference length in scattering by a spherical inhomogeneity is taken to be the radius of the first Fresnel zone, ℓ_F , a quantity which depends on the point of observation relative to the advancing (spherical) wave front, and on the wave length. Spheres of size $\ell < \ell_F$ are designated "small", while those of size $\ell > \ell_F$ are called "large." Inside the Fresnel distance, i.e. $L < L_d \sim \frac{\ell^2}{\lambda}$ the field is dominated by interference effects; outside, by diffraction effects. At a distance $L \sim L_n$ refractive effects dominate all changes of field due to propagation in a random medium.

We next suppose there is only one lens at $\underline{R} = (0, z_1)$ in the path of a steady state plane wave traveling in the positive Z- direction. The total acoustic field $p(\underline{x})$ is normalized to the form $p(\underline{x}) = P(\underline{x}) e^{ikz}$, where $\underline{x} = (\underline{\varsigma}, z)$. The scattered field $p_s(\underline{\varsigma}, z)$ is found by considering the scattering source term in the Helmholtz wave equation to be

$$k^2 \delta m (\underline{x}) \exp \left\{ (\underline{x} - \underline{X}_1^2) / \ell^2 \right\} \tag{31.7}$$

and choosing $\underline{x}$ to be $(\underline{\varsigma}, L)$, and $\underline{X}_1$ to be $(0, z_1)$. By use of the free field Green's function for volume scattering one derives the total field at $\underline{\varsigma}, L$ to be

$$U_1(\underline{\varsigma}, L) \sim 1 + \frac{i \pi^{\frac{1}{2}} k \ell \, \delta m(\ell)}{1 + \frac{2i(L-z_1)}{k\ell^2}} \exp \left[- \frac{\varsigma^2 k^2}{1 + \frac{2i(L-z)}{k\ell^2}} \right] \tag{31.8}$$

in which the incident field is unity, and terms of order $(k\ell)^{-1}$ and ℓ/L have been discarded.

The incremental field $\delta U_1 (\equiv U_1 - 1)$ will depend strongly on the ratio of sizes $L - z_1$ to $k\ell^2$ that is, on the magnitude ℓ and on whether the point of observation $(= L)$ is within or beyond the Fresnel distance ℓ^2/λ . It is first assumed that the spherical inhomogeneity ℓ is larger than the Fresnel radius, i.e. $\ell > \ell_F = (\lambda L)^{1/2}$. By inference the point of observation $(= L)$ is located well within the Fresnel distance $(= \ell^2/\lambda)$ so that $L < k\ell^2$. Then, the incremental field is given by

$$\delta U_1(\underline{\varsigma}, L) \sim \left[ik\ell + 2\frac{(L-z_1)}{\ell} \right] \pi^{1/2} \delta m (\ell) \exp \left\{ - \frac{\varsigma^2}{\ell^2} - 2i(L-z_1)\frac{\varsigma^2}{k\ell^4} \right\} \tag{31.9}$$

If there is a large screen of inhomogeneities, each of size ℓ , at z_1 the oscillating exponential will vanish due to the random nature of the phases. This screen is equivalent to

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a total field $U(z_1)$ in the scattering plane $z = z_1$. Using it one sees that at point $(0, L)$ for all $|\varsigma| \ll \ell$, the incremental (Fresnel) field is

$$\delta U(L) \sim \left[i k \ell + 2(L - z_1)/\ell \right] \delta n(\ell) U(z_1) \tag{31.10}$$

It is next assumed that the size $(= \ell)$ of inhomogeneity is less than the Fresnel radius, i.e. $\ell < \ell_F$. By inference the point of observation L is well beyond the Fresnel distance ℓ^2/λ, (alternatively $L > k\ell^2$). Writing $\theta = |\varsigma|/(L - z_1)$ and using the condition $\ell < \ell_F$ to fix the magnitude of L relative to $k\ell^2$ in Eq. (31.9), one arrives at the Fraunhofer-zone approximation,

$$\delta U_1(\theta, L) \sim \frac{k^3 \ell^3 \delta n(\ell)}{2(L - z_1)} \left[1 + \frac{i k \ell^2}{2(L - z_1)} \right] \exp \left[-\left(\frac{k \ell \theta}{2} \right)^2 - i \frac{k(L - z_1)}{2} \theta^2 \right]$$

Note here that the far-field diffraction pattern lobe, defined by $\theta_d \sim \lambda/\ell \quad (\sim \ell/k\ell^2)$ (31.11a) is much wider than the first (interference) lobe $\theta_F \sim (\lambda/L)^{1/2} = (kL)^{-1/2}$. For a screen of inhomogeneities at z_1 the region in plane $z = L$ over which constructive interference occurs is of radius $(L - z_1)\theta_F$ (i.e. "the first Fresnel zone" $(= (\lambda L)^{1/2})$. Replacing the effect of the screen by a field $U_1(z)$, suppressing the exponentials, and noting that the number of spherical inhomogeneities in the screen at z_1 is $(L - z_1)/k\ell^2$, it is seen that the incremental field at L is given by

$$\delta U(L, \iota) \sim \frac{k_i}{2} \left[1 + i k \ell^2 / 2(L - z_1) \right] \delta n(\ell) U(z) \tag{31.11b}$$

The form of Eqs. (31.10) and (31.11b) immediately suggests the investigation of the ratios $\delta U_i / U$, i.e. a study of the $\ell n\, U$ rather than U itself. Let $\psi = \chi + i\phi = \ell n\, U$, then a <u>small</u> change in $\psi = \delta U/U$. Thus we are led to consider the small complex quantity $\delta \psi$, which is the regime of weak scattering.

Weak Scattering

Here one treats of two cases. In the first let $\ell > \ell_F$. Since $d\psi = d\chi_L + i\, d\phi_L$, (subscript means large) it is seen from Eq. (31.10) that for a string of spheres L/ℓ from $z = z_1$ to $z = L$, one has,

$$\langle d\phi_L^2(\ell) \rangle \sim C_m^2 k^2 L \ell^{2/3} \tag{31.12}$$

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This is the covariance of the random change in phase at observation distance L as a function of a string of spheres of single size ℓ for a medium whose random variation of the index of refraction satisfies a Kolmogoroff type of spectrum. By integrating over all sphere sizes from ℓ_F to L_o, and noting that $\ell_F \sim \left(\frac{\lambda}{L}\right)^{1/2}$, one finds the covariance of the phase change at distance L from the source plane at Z_1 (where $Z_1 \ll \ell$), to be,

$$\langle d\phi_-^2 \rangle \sim C_m^2 k^2 L L_o^{5/3} - \gamma \sigma_\epsilon^2, \qquad \gamma \sim 1 \tag{31.13}$$

where

$$\sigma_\epsilon^2 = C_m^2 k^{7/6} L^{11/6} \tag{31.14}$$

In the second case let $\ell < \ell_F$. Forming the ratio $\delta U/U(z)$ from Eq. (31.11b), multiplying it by the number of spheres $(L-z_1)/k\ell^2$ in the plane Z, and the number of spheres L/ℓ in the string from z_1 to L, then integrating from ℓ_o to ℓ_F, one finally arrives at the approximation

$$\langle d\phi_s^2 \rangle \sim \sigma_\epsilon^2 \tag{31.15}$$

in which the contribution from ℓ_o has been neglected. Thus to within an equality of unity of the symbol γ the covariance of the random phase from scatterers of <u>all</u> sizes at the single point $0, L$ is

$$\langle \delta\phi^2 \rangle \sim C_m^2 k^2 L L_o^{5/3} - \gamma \sigma_\epsilon^2 \tag{31.16}$$

The first term r.h.s. is the geometric optics result (i.e. $(\lambda L)^{1/2} \ll \ell_o$). The second term r.h.s. is the contribution from spheres of all sizes $\ell > \ell_o$, but chiefly from $\ell \sim \ell_F$. Clearly the magnitude of σ_ϵ^2 is of crucial importance in determining the variance of the random phase in weak scattering.

We consider next the correlation of the random phases $d\phi_1, d\phi_2$ at the symmetrical points $\pm |s|/2$ in the plane $z = L$. Let $\phi(\underline{s}, z)$ be the phase function in plane z, and $\delta\phi(K, z)$ be its two dimensional Fourier transform. By noting that $K \sim \ell^{-1}$, and displaying the covariance C_ϕ of $d\phi$ as an 2-dimensional transform in K-space one can deduce that in the plane $z = L$,

$$C_\phi^\ell(\underline{s}) \equiv \langle d\phi_1(0) d\phi_2(\underline{s}) \rangle \sim \langle d\phi^2(0) \rangle J_0(s/\ell) \tag{31.17}$$

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in which $\langle \delta \phi^2(\ell) \rangle$ is the variance of the random phase (i.e. at $|\underline{\varsigma}| = 0$). Since the structure function for random phase is defined as $\Delta C_\phi(\varsigma) \equiv C_\phi(0) - C_\phi(\varsigma)$ it is seen that

$$\Delta_\phi^\ell(\varsigma) \sim \langle \delta \phi^2(\ell) \rangle \left[1 - J_0(\varsigma/\ell) \right], \qquad \varsigma = |\underline{\varsigma}| \qquad (31.18)$$

There are two asymptotic forms, namely when ς/ℓ is very large, and ς/ℓ is very small. These forms are

$$\Delta C_\phi^\ell(\varsigma) \sim C_m^2 k^2 L \ell^{-4/3} \varsigma^2 \qquad\qquad \varsigma < \ell \qquad (31.19a)$$

$$" \sim C_m^2 k^2 L \ell^{2/3} \qquad\qquad \varsigma > \ell \qquad (31.19b)$$

The phase structure function is obtained from $\Delta C_\phi^\ell(\varsigma)$ by integration over all ranges of ℓ, from ℓ_0 to L_0 . The result of this integration depends on the size of ς : for $\varsigma < \ell_0$ one uses Eq. (31.19a); for $\varsigma > L_0$ one uses Eq. (31.19b); and for $\varsigma \sim \ell$ one uses Eq. (31.19a) from $\ell = \varsigma$ to $\ell = L_0$, and Eq. (31.19b) from $\ell = 0$ to $\ell = \varsigma$. The integration results are listed as,

$$\Delta C_\phi(\varsigma) \sim C_m^2 k^2 L \ell_0^{-1/3} \varsigma^2 \qquad\qquad \varsigma < \ell_0$$

$$\sim C_m^2 k^2 L \varsigma^{5/3} \qquad\qquad \ell_0 < \varsigma < L_0 \qquad (31.20)$$

$$\sim C_m^2 k^2 L L_0^{5/3} \qquad\qquad \varsigma > L_0$$

In the range $\ell_0 < \varsigma < L_0$ there is a $\varsigma = \varsigma_\phi$ where $\Delta C_\phi(\varsigma) \sim \pi$. This is taken to be the coherence width in the plane $z = L$, and has the value,

$$\varsigma_\phi \sim \ell_F (\sigma_\epsilon^2)^{-5/3} \qquad (31.21)$$

When $\ell > \ell_F$ the random change in amplitude from z to L is given by the real part of Eq. (31.10) namely $d\chi_L(\ell) \sim [2(L-z)/\ell] \delta m(\ell)$. Multiplying by L/ℓ (= number of spherical lenses in the path), then integrating over the range $\ell = \ell_F$ to $\ell = L_0$ on obtains approximately

$$\langle \delta \chi_L^2 \rangle \sim C_m^2 L^3 \ell_F^{-7/3} \sim \sigma_\epsilon^2 \qquad (31.22)$$

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in which the chief contribution comes from $l \sim l_F$. A similar procedure for the case $l < l_F$ also leads to the conclusion that $\langle \delta \chi_s^2 \rangle \sim \sigma_\epsilon^2$. Thus, one concludes that the variance of the random change of amplitude in the plane $z = L$ is given by,

$$\langle \delta \chi^2 \rangle \sim \sigma_\epsilon^2 \tag{31.23}$$

However, this result is valid only when $\sigma_\epsilon^2 \ll 1$, a condition arising from the requirement that <u>refraction through an inhomogeneity be negligible</u> (i.e. $\langle \theta_n^2 \rangle \ll \theta_F^2$).

To obtain the covariances of amplitude change at (transverse) points $\pm \varsigma/2$ in the plane $z = L$, requires the variances of $\delta \chi(l)$. These are

$$\langle \delta \chi_L^2 (l) \rangle \sim C_m^2 L^3 l^{-10/3} , \qquad\qquad l > l_F$$

$$\langle \delta \chi_s^2 (l) \rangle \sim C_m^2 k^2 L l^{2/3} , \qquad\qquad l < l_F \tag{31.24}$$

The log-amplitude of covariance is obtained (as in the earlier case of random phase) by use of the two-dimensional Fourier transform in the transform pair $K\varsigma \sim \varsigma/l$. The result is similar to the result for phase change, namely, that in the plane $z = l$,

$$C_\chi^l (\varsigma) \sim \langle \delta \chi^2 (l) \rangle J_0 (\varsigma/l) \tag{31.25}$$

Using the asymptotic forms for $J_0 (\varsigma/l)$, and integrating over all sizes of l , one derives the following expressions for the covariance of the random changes in amplitude,

$$C_\chi (\varsigma) \begin{cases} \sigma_\epsilon^2 \left[1 - \gamma \varsigma^2 \Big/ l_0^{\frac{1}{3}} (\lambda L)^{5/6} \right], & \varsigma < l_0 \\[3mm] \sigma_\epsilon^2 \left[1 - \gamma \left(\dfrac{k \varsigma^2}{L} \right)^{5/6} \right], & l_0 < \varsigma < l_F \\[3mm] \sigma_\epsilon^2 \gamma \ (k \varsigma^2 / L)^{-7/6} & \varsigma > l_F \end{cases} \tag{31.26}$$

Note that the <u>main contributions to $C_\chi(\varsigma)$ are from spheres of size $l \sim \varsigma$</u> .

The parameter σ_ϵ^2 (Eq. (31.13)) which appears in the statistics of both random phase and random amplitude is the key quantity in the theory of volume scattering. For weak

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scattering $\overline{J_\epsilon}^2$ must be much less than unity. By defining ψ such that the random field $U = e^\psi$, any probability distribution in random ψ corresponds to a log of this distribution for the random function U. Now the basic assumption in weak scattering is that there are a large number of (weakly) scattering spheres in the range z_i to L. By the central limit theorem one concludes that the random function $\psi = (\chi + i\phi)$ is normally distributed. Hence e^ψ is log-normally distributed. Now when scattering occurs, the central field e^ψ is diminished by losses scattered into off-axis angles of scattering. This weakening of the central field is described by a factor $A_{LN} < 1$ which multiplies e^ψ. The off-axis contribution to the received field is described by a term δU_R. Thus the received field at $z = L$ is given by weakened central field, and by off-axis contributions, i.e.

$$U = A_{LN}\, e^\psi + \delta U_R \tag{31.27}$$

To find the relative contributions to each term of this equation one must consider multiple scattering effects. A good first approximation of the field $\delta U_1(\theta, L)$ is given by Eqs. (31.9) and (31.11a), (where $e = (L - z_i)\theta$). Let $dI_1(\theta) = \langle |\delta U_1(\theta, L)|^2 \rangle$. Then by squaring Eq. (31.9) and Eq. (31.11a) one obtains

$$dI_1(\theta) \quad \begin{cases} I(0)\langle \delta m^2(\ell)\rangle k^4 \ell^6 (L z_i)^{-2} \exp\left[-\left(\frac{k\ell\theta}{2}\right)^2\right], & \ell < \ell_F \\[2mm] I(0)\langle \delta m^2(\ell)\rangle k^2 \ell^2 \exp\left[-2\left(\frac{L\theta}{\ell}\right)^2\right], & \ell > \ell_F \end{cases} \tag{31.28}$$

Since z_i is now a variable, one integrates these equations for all θ, and all z_i (from 0 to L). The result is

$$\frac{dI_1(\ell)}{I(0)} \sim \langle \delta m^2(\ell)\rangle k^2 L\ell \sim C_n^2 k^2 L\, \ell^{2/3} \tag{31.29}$$

If this formula could be integrated over all ℓ, the field intensity at $z = L$ would have the form $I(L) = I(0) e^{-\alpha L}$, with $\alpha L = \frac{dI_1(0)}{I_1(0)}$ where 0 means $\theta = 0$, and $\alpha = \alpha(\ell)$. One thus infers that α can be defined as,

$$\alpha(\ell) = C_n^2 k^2 \ell^{2/3} \tag{31.30}$$

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The contributions to α_s from all spheres $\ell=0$ to ℓ_F then is found by integration,

$$\alpha_s \sim C_m^2 k^2 \ell_F^{5/3} \sim \sigma_\epsilon^2 / L \qquad (31.31)$$

In the regime of weak scattering, since $\sigma_\epsilon^2 \ll 1$, it is seen that $\alpha_s L$ is very small, and hence only single scattering is important. Off-axis contributions in the range $(0, L)$ can thus be ignored in weak scattering. When however the spheres are large $(\ell > \ell_F)$, the integration of $\alpha(\ell)$ given by (31.30) over the range of $\ell = \ell_F$ to L_0 gives

$$\alpha_L \sim C_m^2 k^2 L_0^{5/3} - \sigma_\epsilon^2 / L \qquad (31.32)$$

Thus even if $\sigma_\epsilon^2 \ll 1$ (which is the condition of weak scattering) the wave is still multiple-scattered beyond the distance L_1, where

$$L_1 \sim (C_m^2 k L_0^{5/3})^{-1} \qquad (31.33)$$

Since we choose $L \gg L_1$ it is seen that large spheres ("large" means $\ell > \ell_F$) scatter the wave energy many times. The direction of this scattering is chiefly forward because of the large (acoustic) size of the scatterer.

We return now to Eq. (31.27). It has been inferred that e^ψ is log normally distributed. The term δU_R due to off-axis contributions appears to be <u>Rayleigh distributed</u>. Thus the random field at U is a mixture of log-normal and Rayleigh distributions. The relative proportions between e^ψ and δU_R is accounted for by multiple scattering. The contribution to δU_R is estimated by the ratio

$$\frac{\alpha_s}{\alpha_s + \alpha_L} \sim \frac{\ell_F^{5/3}}{L_0^{5/3}} \sim \left(\frac{L}{k L_0}\right)^{5/6} \qquad (31.34)$$

In the regime of weak scattering this is very small relative to unity. The contribution to e^ψ must be proportional to

$$\frac{\alpha_L}{\alpha_s + \alpha_L} \sim \frac{\ell_F^{5/3}}{L_0^{5/3}} \sim \left(\frac{L}{k L_0^2}\right)^{5/6} \qquad (31.35)$$

For $\sigma_\epsilon^2 \ll 1$ (i.e. in the case of weak scattering) this ratio is approximately unity, and hence the distribution of U is approximately log normal, the off-axis components being negligible.

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The regime of weak scattering, covered by the formulas developed above, is characterized by the requirement that $\sigma_\epsilon^2 \ll 1$. This condition can also be stated in the form $\langle \theta_R^2 \rangle < \theta_F^2$, i.e. the condition that refraction through the spheres be negligible compared to interference and diffraction effects. In the regime of <u>saturation</u> (or the <u>intermediate</u> regime), $\langle \theta_R^2 \rangle > \theta_F^2$. This regime is discussed next.

<div align="center">Saturation Regime</div>

In the saturation regime a sphere at ϱ_1, z_1 (where $\varrho_1 \neq 0$) contributes a <u>refracted</u> field to the point $(0, L)$. To describe this event one uses Eq. (31.6), (repeated here for convenience), (with $= \theta = \frac{\varrho}{(L - z_1)}$)),

$$\delta U_1(\theta, L) \sim \left[ik\ell + 2(L - z_1)/k \right] \delta n(\ell) \exp \left[-\left(\frac{\varrho}{\ell}\right)^2 - \frac{2i(L-z)\varrho^2}{k\ell^4} \right] \tag{31.36}$$

The coordinate ϱ in the plane $z = L$ must now be replaced by the coordinate $\varrho - \varrho_R$ where ϱ_R represents the <u>cumulative effects</u> of refraction in the range z_1 to L. For example, in the absence of refraction a ray at plane z_1 propagates in a straight line to $(0, L)$ whereas in the presence of refraction the same ray propagates to $(-\varrho_R, L)$. The angle of refraction θ_R is given by

$$\theta_R = \frac{\varrho_R}{(L - z_1)} \tag{31.37}$$

This angle is a <u>random</u> variable. Since it related to phase change by the formula

$$\theta_R(\ell) \sim \left[d\phi_1(\ell) - d\phi_2(\ell) \right] / k\varrho \tag{31.38}$$

in which $d\phi_1 - d\phi_2$ is the phase difference between two rays separated by distance ϱ (and measured in the phrase front), it is seen that $\theta_R(\ell)$ is a Gaussian variable with zero mean and variance

$$\langle \theta_R^2(\ell) \rangle \sim \Delta C_\phi^\ell(\varrho) / k^2\varrho^2 \sim C_n^2 L \ell^{-4/3} \tag{31.39}$$

Intergrating ℓ over large spheres $\ell > \ell_F$ on obtains,

$$\langle \theta_R^2 \rangle_{\substack{\text{large} \\ \text{spheres}}} \sim \sigma_\epsilon^2 / kL \tag{31.40}$$

This result comes mainly from spheres of size ℓ_F. A similar integration over small spheres yields the same result, since refraction again dominates. Thus

$$\langle \theta_r^2 \rangle \sim \sigma^2/kL$$

(31.41)

Upon substituting $\langle \theta_r^2 \rangle = \mathcal{E}_r^2/(L-z_i)^2$ in the above equation it is seen that

$$\mathcal{E}_r^2 \sim \frac{\sigma_\epsilon^2 (L-z_i)^2}{kL} \sim \frac{\sigma_\epsilon^2 (L-z)}{k}$$

(31.42)

Again one returns to (31.36) and seeks the conditions under which the log-normal term in (31.27) is most important. This means one seeks the conditions under which the exponential term in (31.32) can be ignored. The requirements are seen to be $\mathcal{E}_r < \ell$, and $\mathcal{E}_r^2 < k\ell^4/L$. One can therefore define a new scale size ℓ_s, where

$$\ell_s \sim \mathcal{E}_r \sim \sigma_\epsilon \left(\frac{L-z}{k} \right)^{1/2}$$

(31.43)

If the sphere size ℓ is such that $\ell > \ell_s$ then the exponential terms in (31.36) can be ignored. The amplitude $\langle \delta \chi_L^2 (\ell) \rangle$ in Eq. (31.24) can be integrated from $\ell = \ell_s$ to $\ell = L_0$. Noting that the main contribution is from ℓ_s, one obtains

$$\langle d\chi_L^2 \rangle \sim C_n^2 L^3 \ell_s^{-7/3} \sim (\sigma_\epsilon^2)^{-1/6}$$

(31.44)

For this to be valid one requires

$$\sigma_\epsilon^2 > 1 \quad ; \quad \ell_s << L_0 \quad ; \quad or \quad \sigma_\epsilon^2 << kL_0^2/L$$

(31.45)

Eq. (31.44) is the variance of the scattered amplitude in the saturation regime for spheres of size $\ell > \ell_s$. This component of U is log-normally distributed. When $\ell < \ell_F$, large _refractive spreading occurs_, and the log-normal component is reduced (through the factor A_{LN}). In this range of sizes of the spheres, one can use (31.30) and (31.31), and, changing the ranges of integration of ℓ to $[\ell_0, \ell_s]$ to obtain α_s'; and to $[\ell_s, L]$, to obtain α_L', it is then seen that

$$\alpha_s' \sim C_n^2 k^2 L \ell_s^{5/3} \quad ; \quad \alpha_L' \sim C_n^2 k^2 L \left(L_0^{5/3} - \ell_s^{5/3} \right)$$

(31.46)

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Thus, the off-axis component of the scattered field is seen to be

$$\langle |\delta U_R|^2 \rangle \sim \left(\frac{l_s}{l_0} \right)^{5/3} \sim \left(\sigma_\epsilon^2 L \Big/ k L_0^2 \right)^{5/6} \tag{31.47}$$

and the log-normal component is

$$A_{LN}^2 \sim 1 - \left(\frac{\sigma_\epsilon^2 L}{k L_0^2} \right)^{5/6} \tag{31.48}$$

The results in Eqs. (31.47) and (31.48) are valid for $\sigma_\epsilon^2 \gg 1$, i.e. they are asymptotic in the saturation regime. By contrast the regime $\sigma_\epsilon^2 \sim 1$, lying in between weak scattering and saturation, is hard to establish. When σ_ϵ^2 is so great that it is comparable (or exceeds) $k L_0^2 / L$ then forward scattering becomes negligible and all the energy goes into the Rayleigh component δU_R. The statistics of intensity are then of the well known Rayleigh type, i.e.

$$\langle I^N \rangle = N! \, I_0^N \tag{31.49}$$

32. Monte Carlo Methods

Ray-tracing methods, widely used in modelling of propagation of acoustic waves in the ocean, ignore all effects due to the finiteness of the wavelength of the sound. If the propagation paths include incidence on rough ocean surfaces and bottoms the ray-tracing method is further limited to the wavelengths much less than the characteristic size L_c of the roughness. Thus the scattering process, being frequency dependent, cannot be directly inserted into a ray-tracing routine for those conditions where the acoustic wavelengths are comparable or greater than L_c. To achieve the very desirable mating of ray-tracing methods and frequency dependent scattering theory one can use the Monte Carlo Method (Schneider [17]).

The following train of thought can be used to illuminate the application of the Monte Carlo method to the scattering of sound from a random surface. An acoustic ray (or wavefront) incident on a random ocean surface at grazing angle φ_0 is scattered into a random oriented outgoing ray. The direction of this ray is given by a scattering coefficient γ, which is dependent on acoustic wavelength λ, roughness scale L_c, and φ_0. For convenience γ is normalized such that

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$$\int_{0}^{180} \gamma(\varphi \mid \varphi_0, L_c, \lambda) \, d\varphi = 1 \qquad (32.1)$$

This unity normalization leads directly to the concept that γ can be interpreted as a probability density for the random angle of scattering φ. This interpretation then leads to the probability function Γ, where

$$\Gamma(\varphi \mid \varphi_0, L_c, \lambda) = \int_{0}^{\varphi} \gamma(\varphi' \mid \varphi_0, L_c, \lambda) \, d\varphi' \qquad (32.2)$$

If φ_0, L_c, λ are given Γ becomes the probability that the scattering angle is less than or equal to φ in an ensemble of trials of projecting incident rays repeated at the same grazing angle φ_0 onto locally different points of the boundary. This interpretation is the key element in the method.

We suppose Γ is known, i.e. one can plot $\Gamma(\varphi)$ versus φ for selected φ_0, L_c, λ. Since the ocean surface is random the angle φ of the outgoing ray is random. Hence φ must be chosen at random. A convenient way of doing this is to select at random a number a_i in the interval $[0, 1]$ specifying it to be equally distributed in this interval. This a_i is labelled the probability that random angles φ are less than or equal to a certain angle φ_i obtained by solving the equation,

$$\Gamma(\varphi_i) = a_i = \int_{0}^{\varphi_i} \gamma(\varphi \mid \varphi_0, L_c, \lambda) \, d\varphi \qquad (32.3)$$

When φ_i is found, its associated scattering coefficient becomes $\gamma(\varphi_i \mid \varphi_0, L_c, \lambda)$. If E_0 is the energy of the incident ray, and $q(\varphi_0, L_c, \lambda)$ is the energy loss

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upon scattering from the boundary, the re-radiated energy of the scattered ray is

$$E_0 \, q(\varphi_0, L_c, \lambda) \; \gamma(\varphi_i, L_c, \lambda) \qquad\qquad (32.4)$$

To summarize: An acoustic ray is assumed incident at an angle φ_0 onto a random ocean surface. To select the angle φ_i of scattering we go to a random number generator designed to deliver a number in the interval [0, 1] (any choice being as probable as any other choice) and make a trial. Let the result of the trial be number a_i' . This number is then inserted into the equation of Γ (φ_i) written above, the solution of which gives the scattering angle φ_i' . By interpretation the angle φ_i is the ensemble averaged angle of scattering resulting from an indefinite number of trials of projecting rays precisely at angle φ_0 onto a locally different points of the surface whose probability density for scattering is the known function γ . For statistical validity it is obvious that the random surface must be statistically homogeneous in space and stationary in time.

In application the energy carried by a single ray is so small that the sum of a few of them at the terminal point of a given range will fluctuate with the number of rays. To obtain statistically valid results, (i.e. minimum deviation from average) one must sum a sufficiently large number of rays (at different angles φ_0), the actual number being dependent on the roughness L_c , and the acoustic frequency λ .

The most difficult part in actual application of this Monte Carlo Method is to construct the angle dependent scattering function γ . A model for γ is formulated and discussed in [17]. In addition, this reference performs Monte Carlo-corrected ray-tracing computations for ocean surface conditions in the North Sea (in mid-winter). A comparison of computed propagation loss of 4001 rays with experimental data shows good agreement between theory and experiment.

The Monte Carlo Method can also be applied to propagation of sound through a medium characterized by stochastic changes of sound speed profiles with ranges. To discuss this we let x, z be the coordinates of range and depth respectively, and ϕ be the propagation angle of a ray (in xz plane). Assume C_0 (z) to be the mean sound velocity profile, and let there be nonstochastic propagation from point x_0, y_0 to x_1, z_1 . The ray path is then given by

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$$(X_1, Z_1, \varphi_1) = f\left(C_0(z); X_0, Z_0, \varphi_0 \right) \tag{32.5}$$

Assume next that there is a random component $\Delta C = \Delta C\, (\, C_0(z\,),z\,)$ in the sound speed profile and let there be stochastic propagation from point X_0, Z_0 to ξ_1, ζ_1. Then the ray path is

$$(\xi_1, \zeta_1, \Phi_1) = (X_1, Z_1, \phi_1) + g\left(C_0(z) + \Delta C; X_0, Z_0, \varphi_0 \right)$$

$$'' \qquad = (X_1, Z_1, \varphi_1) + (\Delta X_1, \Delta Z_1, \Delta \phi_1) \tag{32.6}$$

This means the ray path can be considered a sum of nonstochastic part attributable to propagation through the mean sound speed profile and a stochastic part, attributable the random variations in this profile with range. The random changes in the ray path are depth ΔZ_1, angle $\Delta \phi$, and range ΔX_1. Thus to perform ray-tracing computations one requires a probability density P for the random variables ΔZ, $\Delta \phi$, and ΔX. To simplify applications it is convenient to refer P to a unit range distance $\bar{x}$. The required density is then

$$P\left(\Delta z, \Delta \varphi \mid C_0(z) + \Delta C(z); X_0, Z_0, \varphi_0; \bar{X}, \lambda \right) \tag{32.7}$$

This is the joint probability density that a change ΔZ, and $\Delta \varphi$ of ray path will occur in unit distance from a given starting point X_0, Z_0, and starting angle φ_0.

In application (assuming P is known) one constructs a random number generator based on the P which on any particular trial for given $C_0\,(z)$, $\Delta C\,(z)$, X_0, Z_0, φ_0, $\bar{X}$, λ delivers two numbers ΔZ_1, $\Delta \varphi_1$. A given ray known to pass through point

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χ_0 , z_0 with angle ϕ_0 , is computed by ray-tracing computation on the mean profile at unit range $\chi_1 = \chi_0 + \bar{\chi}$ to pass through z_1 at angle φ_1 . At z_1 the ray is moved in the z-direction an amount $\Delta \phi_1$, and is turned in angle an amount $\Delta \phi_1$. Thus the ray passes through $z_1 + \Delta z_1$ at angle $\phi_1 + \Delta \phi_1$. With these new points on the ray path serving as input parameters the random number generator delivers two new numbers Δz_2 , $\Delta \varphi_2$, and the process of constructing the path is continued in the identical manner as before.

As noted earlier the application of the Monte Carlo Method to sound propagation through a random sound speed profile requires a knowledge of the probability density of the random displacement of the ray path in the z (or vertical) direction, and its random turn in angle φ . In addition a sufficiently large number of rays must be traced to ensure a minimum deviation in fluctuations of energy at a selected receiving plane in the range x. However the number of rays is not the only requirement. It is also necessary to ensure that the spacing of the rays (in angle ϕ and depth z) must be close enough to make the ensemble averaging valid.

Further discussion, with experimental results, will be found in [17].

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33. Ray-Optic Parameters and Ocean Fluctuations (JASON Approach)

In ray-optic theory it is known that the difference in <u>phase</u> $\phi(\underset{\sim}{r}_1) - \phi(\underset{\sim}{r}_2)$ is obtained by a path integral over the wavevector $\underset{\sim}{k}$,

$$\phi(\underset{\sim}{r}_1) - \phi(\underset{\sim}{r}_2) = \int_{r_1}^{r_2} \underset{\sim}{k} \cdot d\underset{\sim}{r} \tag{33.1}$$

(see Macuvitz and Felsen "Radiation and Scattering of Waves" p. 128). Let the component of $\underset{\sim}{k}$ in the direction $d\underset{\sim}{r}$ be $k \mu(s)$, where s is the parametric variable describing the path, $k = 2\pi/\lambda$, and $\mu(s)$ is the index of refraction. Thus

$$\phi(s_1) - \phi(s_2) = k \int_{s_1}^{s_2} \mu(s) ds \tag{33.2}$$

In the ocean $\mu(s)$ is a random variable. Designating the variance of this path integral as Φ^2, one can write,

$$\Phi^2(R) \equiv \left\langle \left(k \int_0^R \mu(s) ds \right)^2 \right\rangle \tag{33.3}$$

Instead of one path from 0 to R we now consider two paths $(1,2)$ starting at $s = 0$ and ending at R_1, R_2. The difference of the path integrals $1,2$ is a random variable. Its variance is the <u>structure function</u> of the (random) path integrals, defined as

$$D(|R_1 - R_2|) = D(y, \zeta, t) = k^2 \left\langle \left(\int_1 \mu(s) ds - \int_2 \mu(s) ds \right)^2 \right\rangle \tag{33.4}$$

in which the components of $|R_1 - R_2|$ are y (the range coordinate) ζ (the depth coordinate) and t (the time coordinate). If separation distance and time between $1,2$ are sufficiently large the product $\sqrt{\Phi_1^2 \Phi_2^2}$ will be small relative to Φ_1^2 or Φ_2^2, i.e. Φ_1^2 and Φ_2^2 will not be correlated. In order to express this condition we first note that one can define a correlation length $\mathcal{L}(\theta, z)$ of fluctuation in the ocean at ray angle θ by the relation,

$$\Phi^2 = k^2 \int_0^R ds \int_0^R ds' \varrho(s-s') = k^2 \int \langle \mu^2(z) \rangle \mathcal{L}(\theta, z) ds(\theta) \tag{33.5}$$

in which $\varrho(s-s')$ is the correlation function of $\mu(s)$. The value of this integral in any particular case can then be used to define a correlation length $\mathcal{L}(0, z)$ at $\theta = 0$, such that

$$\Phi^2(z, R) = k^2 \langle \mu^2(z) \rangle \mathcal{L}(0, z) R \tag{33.6}$$

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*W.H. Munk, F. Zachariasen "Sound Prop. Through a Fluctuating Stratified Ocean" Stanford Res. Inst. JSS 75-1

this is the variance of the fluctuating phase change at depth z and range $\mathcal{R}$ expressed in terms of the variance of the index of refraction at the observation point $(z, \mathcal{R})$, the correlation length $\mathcal{L}(0, z)$ and the range $\mathcal{R}$ itself. In the general case the ocean is not spatially isotropic nor temporally stationery. There are two correlation lengths: the depth (or vertical) length, ℓ_V, and the range (or horzontal) length ℓ_H. There is also the temporal correlation time, τ. Thus for the two selected paths 1,2 there will be a set of correlation lengths for each path. To simplify the calculations we take one path as reference and assume the second path is not too different. One can then write the structure function in the form,

$$D(|\underset{\sim}{R}_1 - \underset{\sim}{R}_2|) \approx 2\,\Phi^2 \left\{ \left(\frac{\zeta}{\ell_V}\right)^2 + \left(\frac{y}{\ell_H}\right)^2 + \left(\frac{t}{\tau}\right)^2 \right\} \tag{33.7}$$

A strict derivation of this formula can be obtained by noting that D is positive definite, and must vanish for zero separation of field points 1,2. The form has the following merit. First, the average path (= reference path) is defined in terms of a depth z and range $\mathcal{R}$ variance of phase difference Φ^2, and an reference correlation length $\mathcal{L}'(0, z)$. Second, the structure function vanishes when the separations ζ, y, t are all small relative to their respective correlation lengths. This condition means that the phases at the terminal of 1,2 are well correlated. In contrast when ζ, y, t are large relative to ℓ_V, ℓ_H and τ the structure function becomes large, meaning that the phases at the termine 1,2 are uncorrelated. Evidently the structure function D and its associated function Φ^2 is an environmental parameter of fundamental importance in the investigation of the phase statistics of acoustic signals in the ocean.

The environmental parameter Φ^2 essentially deals with the refractive properties of the ocean. It is equally important to construct a parameter that deals with the diffractive properties of the medium. This is Λ. To define Λ we turn to the theory of Fresnel diffraction. Let an undiffracted ray proceed from source to receiver, and let a second ray start from the same source, proceed some distance X, then be bent through an angle θ, finally reaching the receiver. If the separation between the first and second rays at X is such that the second ray has a path length one half wavelength larger than the unperturbed ray, then the separation distance is $\mathcal{R}_F$, the radius of the first Fresnel zone.

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Now if L is the length of the unperturbed ray, then from ray-optic theory it is easily shown that $R_F = (\lambda L)^{1/2}$. Hence, $\lambda = R_F^2/L$. From the theory of diffraction the angle θ_d through which a ray is bent by the presence of an obstacle of size ℓ is approximately $\theta_d \sim \lambda/\ell$. Thus, $\theta_d \approx R_F^2/\ell L$. It is convenient to take the length L to be the scale of fluctuations ℓ. The angle of diffraction (normalized to 2π) then becomes ,

$$\Lambda \equiv \frac{\theta_d}{2\pi} \approx \frac{1}{2\pi}\left(\frac{R_F}{\ell}\right)^2 \tag{33.8}$$

The magnitude of Λ measures the importance of diffraction in the propagation of signal through inhomogeneities in the ocean.

The relative effects of refraction, diffraction, and ocean fluctuations on the received acoustic signal can be portrayed on a $\ln\Phi$ versus $\ln\Lambda$ graph (= JASON GRAPH). We reproduce below a synoptic account of the types of perturbations on received signals caused by ocean fluctuations. In each case the received signal is a phasor $\psi = Ae^{i\phi}$ of amplitude A, phase ϕ, which is a sum of all acoustic energies reaching the field point at specific time after transit over a range R (or equivalent time t). The phasor oscillates (or rotates) about a fixed point, and the path traced by its tip is intended to indicate time variation of the signal at the receiver, both as regards its amplitude and its phase.

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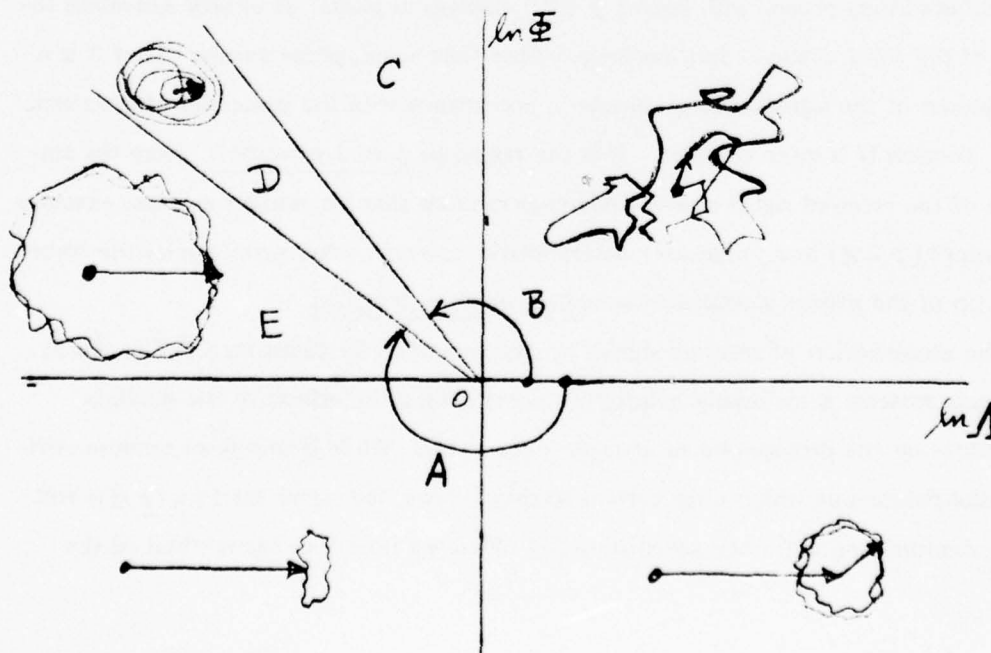
The first quadrant (when both Φ and Λ are greater than unity) characterizes received signals that are random, either because of random multipath, or strong scattering. The Cartesian components of the phasor execute random walking, and the magnitude of the two components is a random variable obeying Rayleigh statistics. The ocean medium randomizes the signal to the point of "saturation," by which is meant the maximum possible randomization consistent with scale sizes of ocean fluctuations and wavelength of the signal (see Sec. 31, Eqs. 31.36 thru 31.49). The third quadrant (where both Φ and Λ are less than unity) exhibits the opposite extreme to the first quadrant. Here the received signal is quite steady in amplitude (as a function of time) while its phase fluctuates over <u>small</u> phase angles (much less than π radians). This is the region of weak, narrow angle scattering (see Sec. 31, Eqs. 31.1 thru 31.35 for further discussion). It corresponds to propagation in accordance with the approximations of geometric optics. In the fourth quadrant (where Λ is greater than unity, but Φ is less than unity) the received signal undergoes moderate but rapid changes in the amplitude, and behaves as a constant *vector tipped by a random vector rotating at irregular angular velocity. The second* quadrant (where Φ is greater than unity, but Λ is less than unity) is partitioned into three subsections. In section C the received signal has the same appearance as that of saturated signals in the first quadrant. In section E the received signal has the appearance of a constant amplitude phasor with large ($> 2\pi$) changes in phase. It closely resembles the signals of the third quadrant but has large, rather than small, phase swings. Thus it is a continuation of the signals that propagate in accordance with the principles of geometric optics. Section D is quite complex. It is the region of <u>partial saturation</u>. Here the amplitude of the received signal phasor undergoes random changes, while the phase executes large swings ($> 2\pi$) in a progressive deterministic manner. After many cycles the trace of the tip of the phasor is described as <u>coiled</u>, or <u>phase-wrapped</u>.

The classification of received signals by assignment to locations on a $\ln\Phi$ vs $\ln\Lambda$ coordinate systems is materially helpful in understanding the effects of the medium fluctuations on the propagation of acoustic phenomena. While Φ and Λ are random environmental parameters which obey certain statistical laws, the signal itself ($\wp(\underset{\sim}{r},t)$) will also be random, and will obey signal statistics. First we note that the method of the

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JASON GROUP is that of ray-optic theory. Hence the phase and amplitude change along a ray is modeled in the following way,

$$p(\underline{r}_2) \sim p(\underline{r}_1) \left[\frac{\mu(\underline{r}_2) dA(\underline{r}_2)}{\mu(\underline{r}_1) dA(\underline{r}_1)} \right] exp \left[i \int_{r_1}^{r_2} \underline{k} \cdot d\underline{r} \right] = A e^{i\phi} \qquad (33.9)$$

It is assumed that both A and ϕ are independent Gaussian random variables. In the geometric optics regions of the $\ln \Phi$ vs $\ln A$ coordinate chart (namely region A) the amplitude of the received signal phasor is quasi constant. Taking this amplitude as unity, we can write the received signal as

$$p(\underline{r}) = exp \left\{ i k \int \mu ds \right\} \qquad (33.10)$$

Since μ is random the signal p is random. We desire the expected (or mean) value of p. Assuming $\int \mu ds$ is a Gaussian random variable it can be shown that

$$\langle p(R) \rangle = exp \left\{ -\frac{\overline{\Phi}^2}{2} \right\} \qquad (33.11)$$

(see Middleton "Introduction to Statistical Information Theory," page 336). Furthermore, at two field points $\underline{r}_1$, $\underline{r}_2$, the statistics are given by,

$$\langle p(\underline{r}_1) p^*(\underline{r}_2) \rangle = exp \left\{ -\frac{1}{2} D(1,2) \right\} \qquad (33.12)$$

$$\langle (\Phi_2 - \phi_1)^2 \rangle = D(1,2) \qquad (33.13)$$

In addition if we allow A to vary, the variance of $\ln A$ is related to Λ and $\overline{\Phi}^2$ as follows:

$$\langle (\ln A)^2 \rangle - \langle \ln A \rangle^2 \approx \Lambda \, \overline{\Phi}^2 \qquad (33.14)$$

Similar signal statistics can be constructed for the other regions of the $\ln \Phi$ vs $\ln A$ chart. In conclusion: The procedures of the JASON GROUP have served to emphasize the following: the real ocean is quite different from being an isotropic, homogeneous, random medium. Salient points of difference are: (1) the length scale of ocean sound speed fluctuation in the depth (i.e. vertical) coordinate is much smaller than the length

scale in range (i.e. horizontal) coordinate. In fact the ocean exhibits a correlation length of blob size which is a function of ray angle of elevation above the horizontal, (2) all environmental parameter of importance are strong functions of depth, (3) there is in the ocean a background sound channel in which propagating rays travel in curved paths rather than instraight lines, (4) the spatial (or wavenumber) spectrum of environmental inhomogeneities is that of internal waves, viz k^{-2}, rather than that of Kolmogorov turbulence, $k^{-5/3}$.

The methods of calculating the effects of ocean fluctuations on acoustic signals must give prominant importance to these differences.

34. Multiple Scattering, Geometrical Optics and the Parabolic Wave Equation

The description of wave propagation in a random medium by concepts of geometrical optics theory is valid when the wavelength λ is much smaller than the characteristic length L_c of the inhomogeneities. An additional requirement is that the radius of the first Fresnel zone must be small relative to the smallest L_c of the random medium. When the range is so great that multiple scattering must taken into account one can use geometrical optics theory to construct a model which includes multiple scattering (see ref. 14, page 166ff).

We begin with Eq. 16.4 of this report and assume a point source of waves in an unbounded homogeneous (i.e. nonstratified) medium. The infinite pertubation series then is written in terms of the Green's function $G(\underset{\sim}{r}, \underset{\sim}{R})$,

$$u(\underset{\sim}{r}) = \sum_{m=0}^{\infty} u_m(\underset{\sim}{r}), \qquad u_0(\underset{\sim}{r}) = G_0(\underset{\sim}{r} - R) = \frac{-\exp\{ik|\underset{\sim}{r} - \underset{\sim}{R}|\}}{4\pi |\underset{\sim}{r} - \underset{\sim}{R}|}$$

(34.1)

$$u_m(\underset{\sim}{r}, \underset{\sim}{R}) = (-k^2)^m \underbrace{\int \cdots \int}_{m} G_0(\underset{\sim}{r} - \underset{\sim}{r}_1) G_0(\underset{\sim}{r}_1 - \underset{\sim}{r}_2) \cdots G_0(\underset{\sim}{r}_m - \underset{\sim}{R})$$

$$\times \tilde{m}(\underset{\sim}{r}_1) \tilde{m}(\underset{\sim}{r}_2) \cdots \tilde{m}(\underset{\sim}{r}_m) d^3r_1 d^3r_2 \cdots d^3r_m .$$

Noting that the integration over two successive Go's is the integral over the commons point (say r_k), we concentrate attention on two paths that begin and end at common points: path 1=A= $|r_{k-1} - r_{k+1}|$, and path 2= $|\underset{\sim}{r}_{k-1} - \underset{\sim}{r}_k| + |r_k - \underset{\sim}{r}_{k+1}|$ = B+C written as function of r_k. Then the k'th integral in Eq. (34.1) has the form

$$I_k = \frac{1}{16\pi^2} \int \frac{e^{ik[B+C]}}{BC} \tilde{m}(\underset{\sim}{r}_k) d^3r_k$$

(34.2)

If B+C is set equal to A=const. then $\exp ik[B+C]$ does not oscillate. This is the case of no scattering. We next suppose B+C is a function of $\underset{\sim}{r}_k$ ($\underset{\sim}{r}_{k-1}$ and $\underset{\sim}{r}_{k+1}$ being fixed) and find the locus of all points r_k such that

$$B(\underset{\sim}{r}_k) + C(\underset{\sim}{r}_k) = \frac{a}{k} \qquad\qquad , a = \text{const.}$$

This is an ellipsoid of revolution with semi major axis $\frac{a}{2k}$, semiminor axis $\sqrt{a^2 - k^2A^2}$ both clearly functions of a. Two values of a are significant in a first approximation.

When $a = a_o = kA$, the semiminor axis collapses to zero and the propagation is in a straight line from $\underset{\sim}{h}_{K-1}$ to $\underset{\sim}{h}_{K+1}$ i.e. (no scattering). When $a_1 = a_o + \pi$, the exponential is multiplied by -1. The semi-minor axis for the choice $a = a_1$, is $(1/2k)\sqrt{2a_o\pi + \pi^2}$. If A is so large that $h_k + 1$ is in the far field of radiation from $\underset{\sim}{h}_k$ then $a_o \gg \pi$ and the semiminor axis is approximately $\rho \approx \sqrt{a_o\,\pi/2k^2} = (1/2)\sqrt{\lambda A}$. If one choose $a_m = a_o + m\pi$ then there results a series of ellipsoids of revolution with semiminor axis $\rho \approx (1/2)\sqrt{m\lambda A}$. At the coordinate of the midpoint of A the distance between successive ellipsoids is about ρ_1. Fig. 34.1 shows the salient features of the family of confocal ellipsoids of revolution. Returning to Eq. 34.2 it is seen that a volume integration over $d h_k^3$ is required. Since the integral contains $\tilde{n}(\underset{\sim}{h}_K)$ it is pertinent to ask how rapidly this factor is changing. We make the important assumption that the characteristic length L_c of the inhomogeneity is much larger than a wavelength. Thus $\tilde{n}(\underset{\sim}{h}_k)$ hardly changes over a distance of a wavelength. Furthermore the additional assumption is made that ρ_1 is much smaller than L_c (i.e. $\sqrt{\lambda A} \ll L_c$), that is, the radius of the first Fresnel zone is much smaller than the characteristic length of the inhomogeneity. With there two assumptions in mind we construct planes D and E through $\underset{\sim}{h}_{K-1}, \underset{\sim}{r}_{K+1}$ respectively, perpendicular to A. The intersection of these planes with the ellipsoids of revolution form a set of concentric circles, which are (by construction) half-wave, or Fresnel zones. We use this picture to help evaluate the volume integral of Eq. 34.2. First, it is noted that outside D, E the integral must vanish since its phase oscillates rapidly over the rest of half-wave zones. Second, the volume integral inside the parallel planes (D, E) will be over those waves which are scattered at an angle less than $\theta \sim \lambda/L_c$ (see Sec. 31 for a discussion of DeWolf's model which features this point), the waves reaching h_{K+1} will at the outer most have arrived from points off the path A (in the yz plane) a distance $\theta A \sim \dfrac{\lambda A}{L_c} = (\sqrt{\lambda A}\,\dfrac{\sqrt{\lambda A}}{L_c})$. But $\sqrt{\lambda A}/L_c \ll 1$, which is the restriction set on the length L_c relative to the radius of of the first Fresnel zone. Thus the volume integral between D, E is restricted to the first few ellipsoids (effectively the first ellipsoid).

In performing the volume integration over the first ellipsoid one can obtain a great simplification by noting that the ellipsoid is much elongated along A (call this the x-direction), making the transverse dimensions (call these y, z) small. Hence B,C can be expanded in a series,
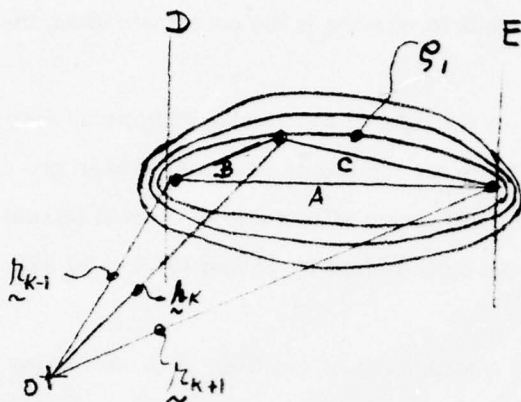
$$\begin{Bmatrix} B \\ C \end{Bmatrix} \approx |x_{k_{\mp 1}} - x_k| + \frac{(y_{k_{\mp 1}} - y_k)^2 + (z_{k \mp 1} - z_k)^2}{2\,|x_{k_{\mp 1}} - x_k|} - \cdots, \qquad (34.3)$$

Retaining the two terms of B, C in the phase exponential of Eq. 34.2, but only one term in the denominator product BC, and performing the integration over y, z one obtains the approximation to the k'th integral,

$$I_k \approx \frac{G_{\mathfrak{f}}\,(\underset{\sim}{r}_{k-1} - \underset{\sim}{r}_{k+1})}{2ik} \int_{x_{k-1}}^{x_{k+1}} \tilde{n}\,(x_k)\,dx_x \qquad (34.4)$$

Here, the variation of $\tilde{m}\,(\underset{\sim}{r}_k)$ with y, z is assumed neglible, relative to the variation along x_k. The Green's function $G_{\mathfrak{f}}$ has the special form,

$$G_{\mathfrak{f}}\,(x, y, z) = -\frac{1}{4\pi\,|\underset{\sim}{x}|}\,\exp\left\{ik\,|\underset{\sim}{x}| + \frac{y^2 + z^2}{2\,|\underset{\sim}{x}|}\right\} \qquad (34.5)$$

With these formulas in mind one can evaluate $U_m(\underset{\sim}{r}_0)$ to be,

$$U_m\,(\underset{\sim}{r}_0) \approx (-k^2)^m\,\frac{G_0\,(\underset{\sim}{r}_0 - \underset{\sim}{R})}{(2ik)^m} \int_{x_0}^{x} dx_m\,\tilde{n}\,(x_m, 0, 0) \int_{x_0}^{x_m} dx_{m-1}\,n\,(x_{m-1}, 0, 0)$$
$$\cdots \int_{x_0}^{x_2} dx_1\,\tilde{n}(x_1, 0, 0)$$

$$" = G_0\,(\underset{\sim}{r}_0 - \underset{\sim}{R})\,\frac{1}{m!}\left[\left(\frac{ik}{2}\right) \int_{x_0}^{x} \tilde{n}\,(x, 0, 0)\,dx\right]^n \qquad (34.6)$$

Thus the sum of all U_m leads to formula for the total field at $\underset{\sim}{r}$, viz.

$$u(\underset{\sim}{r}) = G_0\,(\underset{\sim}{r}_0 - \underset{\sim}{R})\,\exp\left\{\frac{ik}{2}\int_{x_0}^{x}\tilde{n}\,(x, 0, 0)\,dx\right\} \qquad (34.7)$$

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This is a modified geometric optics-type formula in which x is the coordinate along the path.

The above development can be viewed in the light of the parabolic approximation reviewed in Sec. 21 of this report. Clearly there are significant relations between geometric optics, the parabolic equation and the diffraction effects due to random inhomogeneities in the medium. These relations have been derived by Palmer (JASA. $\underline{60}$ 343 (1976)), and are discussed below.

In the ocean the speed of sound $C(\underset{\sim}{x})$ is a function of position. It is convenient (though not physically significant) to write this speed in terms of a mean speed $\overline{C}(z)$ expressed solely as a function of depth, and a range dependent correction $\delta C(\underset{\sim}{x})$. Using this form of $C(\underset{\sim}{x})$ one arrives at the basic inhomogeneous Helmholtz equation,

$$\left[\nabla^2 + \left(\frac{\omega}{\overline{C}(z)}\right)^2 + V(x)\right] \not{p}(\underset{\sim}{x}) = -\delta^3\left(\underset{\sim}{x} - \underset{\sim}{s}\right) \tag{34.8}$$

in which $\underset{\sim}{x}$ is the observation point and $\underset{\sim}{s}$ is the source point. The boundary conditions of the real ocean exhibit range-dependent depth. However, for purposes of understanding the effect of $V(\underset{\sim}{x})$ on the propagation of sound waves the assumption is made that the ocean is a single layer of uniform depth. One can then begin the solution of (34.8) by expanding the pressure field at $\underset{\sim}{x}$ in a single set of normal-mode eigen functions,

$$\not{p}(\underset{\sim}{x}) = \sum_{m,n} A_{mn}(\hat{x}, \hat{s}) Z_n(z) Z_m(z_s) \tag{34.9}$$

in Which $\hat{x} = (x,y)$, $\hat{\underset{\sim}{s}} = (x_s, y_s)$, and $Z_m(z)$ is an eigen function of the reduced operator $\frac{d^2}{dz^2} + \left(\frac{\omega}{\overline{C}(z)}\right)$ with eigen value λ_m. Substitution of (34.9) into (34.8) and using the orthonormality property of Z_m leads to an infinite set of equations in the unknown A_{mm} coupled to each other through $V(\underset{\sim}{x})$,

$$\left(\hat{\nabla}^2 + \lambda_m^2\right) A_{mm}(\hat{x}, \hat{s}) = -\delta_{mm}\delta^2(\hat{x} - \hat{s}) - \sum_{m\ell} M_{m\ell}(\hat{x}) A_{mm}(\hat{x}, \hat{s}) \tag{34.10}$$

$$M_{m\ell}(\hat{x}) = \int dz\, Z_m(z) V(\underset{\sim}{x}) Z_m(z) \tag{34.11}$$

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This formulation closely parallels the theoretical developments of Sec. 24 on the subject of random waveguides. Here $M_{m\ell}$ expresses the cross coupling between modes m and m caused by the range dependent increment δc. Fourier transforming $\hat{x}$ to $\hat{q}$ in A_{mm}, and $\hat{x}$ to $\hat{k}$ in $M_{m\ell}$ results in a set of coupled integral equations in $A_{mm}(\hat{q},\hat{s})$,

$$A_{mm}(\hat{q},\hat{s}) = \frac{\delta_{mm}\exp(-i\hat{q}\cdot\hat{s}) + \sum_\ell \int \frac{d\hat{k}}{(2\pi)^2} M_{m\ell}(\hat{k}) A_{\ell m}(\hat{q}-\hat{k},\hat{s})}{\hat{q}^2 - \lambda_m^2 - i\epsilon} \tag{34.12}$$

(Note the term $i\epsilon$ is inserted to insure convergence of the field at infinity). This set is solved in term by constructing from it a Neumann series (i.e. the r.h.s. of 34.12 is repeatedly inserted into $A_{\ell m}$ of the r.h.s.). The construction of this series changes the $\sum \int$ operations of (34.12) into $\sum \int \prod$ operations, and because of the $\hat{q}-\hat{k}$ convolution dependence also introduces factors of the form $(\hat{q} - k_1 - k_2 - \cdots - \hat{k}_p)^2$ in the denominator of the newly constructed $\sum \int \prod$ terms in the Neumann series. Now in the geometric optics approximation the magnitude of the wave vector $\hat{q}$ of the incident wave is very much larger than the magnitude of the transverse perturbation $\hat{k}$ introduced by scattering from $V(\chi)$. Hence these forms can be written $(\hat{q} - \hat{k}_1 - \cdots \hat{k}_p)^2 \approx \hat{q}^2 - 2\hat{q}\circ[\hat{k}_1 + \hat{k}_2 \cdots + \hat{k}_p] + k_1^2 + k_2^2 + \cdots + k_p^2 -$ cross terms in $\hat{k}_i \cdot \hat{k}_j$. Neglect of the cross-terms is called the supereikonal approximation. The integrals in the Neumann series constructed with the supereikonal approximation can be simplified and evaluated by use of the parametric techniques of Feynman and Schwinger. First, $A_{mm}(\hat{q},\hat{s})$ is inverse Fourier transformed back to $A_{mm}(\hat{x},\hat{s})$, i.e.

$$A_{mm}(\hat{x}) = \int \frac{d\hat{q}}{(2\pi)^2} \exp(i\hat{q}\cdot\hat{x}) A_{mm}(\hat{q}) \tag{34.13}$$

This formula leads to the requirement of evaluating an integral of the form

$$\xi = \int \frac{d\hat{q}}{(2\pi)^2} \frac{e^{i\hat{q}\cdot(\hat{x}-\hat{s})}}{(\hat{q}^2-\lambda_m^2-i\epsilon)[(\hat{q}-\hat{k}_1)^2-\lambda_1^2-i\epsilon]\cdots[(\hat{q}-\hat{k}_1-\hat{k}_2\cdots\hat{k}_p)^2-\lambda_m^2-i\epsilon]} \tag{34.14}$$

Because of the denominator in the integral this is difficult to integrate. By the use of Feynman's formula the reciprocal of this equation's denominator to order p (in the $\hat{k}_p$)

is transormed into a p'th order multiple integral in which the variables of integration are the ordering parameter $\beta_1, \beta_2, \cdots \beta_p$. The integrand of this multiple integral appears as $\frac{1}{a}$. This $1/a$ is next replaced by Schwinger's representation $1/a = i \int_0^\infty d\tau \, e^{-ia\tau}$. The denominator is thus replaced by infinite integrals, at the price of introducing two new parameters β and τ. Next an integration over $\hat{q}$ in the inverse Fourier transform is easily performed. Finally the modal coefficient $A_{nm}(\hat{x}, \hat{s})$ take on the form

$$A_{nm}(\hat{x}, \hat{s}) = \frac{1}{4\pi} \int_0^\infty \frac{d\tau}{\tau} \exp\left\{ i\left[\left(\frac{\hat{x}-\hat{s}}{2\tau}\right)^2 + \lambda_m^2 + i\epsilon\right]\right\} \left(\delta_{nm} + i\tau \int_0^1 d\beta_1 \, U_{nm}(\beta,\tau)\right.$$

$$+ \cdots + (i\tau)^p \sum_{\ell_1 \cdots \ell_{p-1}} \int_0^1 d\beta_1 \int_0^{\beta_1} d\beta_2 \cdots \int_0^{\beta_{p-1}} U_{m\ell_1}(\beta,\tau) \cdots U_{\ell_{p-1}m}(\beta_p,\tau) + \cdots)$$

$$\text{(34.15)}$$

$$'' = \frac{1}{4i\pi} \int_0^\infty \frac{d\tau}{\tau} \exp\left\{ i\tau\left[\left(\frac{\hat{x}-\hat{s}}{2t}\right)^2 + \lambda_m^2 + i\epsilon\right] T_\beta \left[\exp\left(i\tau \int_0^1 d\beta \, U_{mm}(\tau,\beta)\right)\right]\right.$$

in which U_{mm} is given by,

$$U_{mm}(\beta,\tau) = \exp\left[-i\tau\beta(\lambda_m^2 - \lambda_m^2)\right] \int_0^1 dz \, Z_m(z) V(z/\beta,\tau) Z_m(z)$$

$$\text{(34.16)}$$

$$V(z,\beta,\tau) = \int \frac{d\hat{k}}{2\pi} V(z,\hat{k}) \exp\left\{ i\hat{k}\cdot[\hat{s}+\beta(\hat{x}-\hat{s})] + i\hat{k}^2 \tau(\beta^2 - \beta)\right\}$$

and T_β is the Dyson ordering operator. For a brief discussion of the related Dyson question see Sec. 18 of this report where the symbol Q appears as the analog of T_β. Physically the various ordered terms in T_β of (34.15) represents orders of scattering. For example, when $V(\underline{x})$ is negligible, only the δ_{mm} (or zeroth order) term survives. Then it is easy to show that $A_{mm}(\hat{x})$ is the farfield Hankel function. Similarly, by calculating $V(z,\beta,\tau)$ of Eq. (34.16) with omission of $\hat{k}\tau(\beta^2-\beta)$, and using the first term of the series in Eq. 35.15 one arrives at the eikonal approximation in the form of a geometric optics path integral over the index of refraction (see Sec. 16) in which β is the path parameter. Other specifications of β-orderings are in common use. A typical one is the two-point β-ordering which appears is the bilocal approximation (see Eq. 18.19). Another useful approximation is the one-point β-ordering (= 1st order term) of geometric optics plus a stationary phase solution of Eq. (34.15). The result is

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$$A_{nm}(\hat{\underset{\sim}{x}}, \hat{\underset{\sim}{s}}) \sim \frac{1}{4\pi} \left[\frac{2\pi i}{|\hat{\underset{\sim}{x}} - \hat{\underset{\sim}{s}}| \lambda_m} \right]^{\frac{1}{2}} exp \; i\lambda_m |\hat{\underset{\sim}{x}} - \hat{\underset{\sim}{s}}| exp \left[\frac{i |\hat{\underset{\sim}{x}} - \hat{\underset{\sim}{s}}|}{2\lambda_m} U(|\hat{\underset{\sim}{s}} - \hat{\underset{\sim}{s}}|) \right]_{nm}$$

$$U_{k\ell}(|\hat{\underset{\sim}{x}} - \hat{\underset{\sim}{s}}|) = \int_0^1 d\beta \; M_{k\ell} \left[\hat{\underset{\sim}{s}} + \beta (\hat{\underset{\sim}{x}} - \hat{\underset{\sim}{s}}) \right] exp \left[\frac{-i |\hat{\underset{\sim}{x}} - \hat{\underset{\sim}{s}}| \beta}{2\lambda_m} (\lambda_k^2 - \lambda_\ell^2) \right]$$

(34.17)

This formula allows mode coupling (through $V(\underset{\sim}{x})$). Other approximate β-orderings are possible. However it is to be noted that Eq. 34.15 is a functional equation and is intractable in the general case of all orders of scattering. This is the chief difficulty in the method of β-ordering.

Assuming the modal coefficients $A_{mn}(\hat{\underset{\sim}{x}}, \hat{\underset{\sim}{s}})$ as given by (34.15) are available one can substitute them formally into Eq. (34.9), with the understanding that the difficult of β-ordering still remains. A significant transformation of the resultant equation can be obtained by introducing the sum $K(z, z'; \beta) = \sum_m Z(z) Z(z') exp \; i\beta \lambda_m$, and writing,

$$\phi(\underset{\sim}{x}) = \frac{1}{4\pi} \int_0^\infty \frac{d\tau}{\tau} \; exp \left\{ i\gamma \left[\left(\frac{\hat{\underset{\sim}{x}} - \hat{\underset{\sim}{s}}}{2\tau} \right)^2 + i\epsilon \right] \right\} \mathcal{K}^\tau(z, z_s; \tau)$$

(34.18)

in which $\mathcal{K}^\tau$ is effectively the sum over the ordering terms β, obtained by the trick of summing over the normal-modes. However the kernel $\mathcal{K}^\tau$ must in turn satisfy an integral (or corresponding differential) equation itself,

(34.19a)

$$\mathcal{K}^\tau = K + i \int_0^{\tau'} d\tau'' \int_0^L dz' K(z, z'; \tau' \tau'') V(z'; \frac{\tau}{\tau}, \tau) \mathcal{K}^\tau(z, z_s; \tau')$$

$$-i \frac{\partial}{\partial \tau} \mathcal{K}^\tau = \left[\frac{\partial}{\partial z^2} + \left(\frac{\omega}{c(z)} \right)^2 + V(z; \frac{\tau'}{\tau}, \tau) \right] \mathcal{K}^\tau$$

(34.19b)

Thus the summation achieved by (34.18) is purely formal since there is no available solution of (34.19) as it stands.

However, in lieu of attempting to solve Eq. (34.19) one can construct a definition of the kernel $\mathcal{K}^\tau$. A usable definition is of the form

$$\mathcal{K}^\tau(z, z_s; \tau') = \psi(\hbar, z; \tau') \; exp \; i k_0^2 \tau \hbar / |\hat{\underset{\sim}{x}} - \hat{\underset{\sim}{s}}|$$

(34.20)

in which $\hbar = \tau'/\tau |\hat{\underset{\sim}{x}} - \hat{\underset{\sim}{s}}|$ is the rescaled range from source to receiver. With this form of $\mathcal{K}^\tau$, Eq. (34.18) then reduces to

$$\phi(\underset{\sim}{x}) = \frac{1}{4\pi} \int_0^\infty \frac{d\tau}{\tau} \; exp \left\{ i\tau \left[\left(\frac{\hat{\underset{\sim}{x}} - \hat{\underset{\sim}{s}}}{2\tau} \right)^2 + k_0^2 + i\epsilon \right] \psi(\hbar, z; \tau) \right\}$$

(34.21a)

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in which,

$$-i \frac{|\hat{\underset{\sim}{x}}-\hat{\underset{\sim}{s}}|}{\tau} \frac{\partial \psi}{\partial \hbar} = \frac{1}{4\pi} \int_0^\infty \frac{d\tau}{\tau} exp\left\{ i\tau\left[\left(\frac{\hat{\underset{\sim}{x}}-\hat{\underset{\sim}{s}}}{2\tau}\right)^2 + k_o^2 + i\epsilon\right]\right\} \psi(\hbar, z; \tau) \quad (34.21b)$$

Note that k_o^2 has been made quadratic to permit a stationary phase approximation to the integral. Upon making this approximation one obtains the parabolic equation in ψ,

$$-2i k_o \frac{\partial \psi(\hbar, z)}{\partial \hbar} = \left\{ \frac{\partial^2}{\partial z^2} + \left(\frac{\omega}{C(z+\hat{\underset{\sim}{s}}+\hbar\hat{\underset{\sim}{x}})}\right)^2 - k_o^2\right\} \psi(\hbar, z) \quad (34.22a)$$

and

$$p(\underset{\sim}{x}) = \frac{1}{4\pi}\left(\frac{2\pi i}{|\hat{\underset{\sim}{x}}-\hat{\underset{\sim}{s}}|k_o}\right)^{\frac{1}{2}} exp\left(i |\hat{\underset{\sim}{x}}-\hat{\underset{\sim}{s}}|\right) \psi(\hat{\underset{\sim}{x}}-\hat{\underset{\sim}{s}}, z) \quad (34.22b)$$

Eqs. 34.22 constitute the (now standard) parabolic equation evaluated at the stationery point $\tau_c = \frac{|\hat{\underset{\sim}{x}}-\hat{\underset{\sim}{s}}|}{2k_o}$. But, at this point a severe difficulty occurs in the choice of k_o in Eq. (34.20). If the range dependent $C(\underset{\sim}{x})$ is replaced by $C(z)$ in Eq. 34.21b (or 34.22a) and the equations integrated, the result is

$$\psi(\hbar, z; \tau) = exp\left[\frac{i\tau\hbar}{|\hat{\underset{\sim}{x}}-\hat{\underset{\sim}{s}}|}\left(\frac{\partial^2}{\partial z^2} + \left(\frac{\omega}{C}\right)^2 - k_o^2\right)\right] \psi(0, z) \quad (34.23)$$

Choosing τ to be the stationary-phase point τ_c, one has,

$$\psi(|\hat{\underset{\sim}{x}}-\hat{\underset{\sim}{s}}|) = exp\left[\frac{i|\hat{\underset{\sim}{x}}-\hat{\underset{\sim}{s}}|k_o}{2k_o^2}\left(\frac{\partial^2}{\partial z^2} + \left(\frac{\omega}{C}\right)^2 - k_o^2\right)\right] \psi(0, z) \quad (34.24)$$

In the degenerate case the operator $\frac{\partial^2}{\partial z^2} + \left(\frac{\omega}{C}\right)^2$ has one eigenvalue which can be taken to be k_o, in which case $\psi(|\hat{\underset{\sim}{x}}-\hat{\underset{\sim}{s}}|)$ does not vary much at the stationary point. However, in the ocean the sound speed C is a function of depth and range and the stationary phase solution with a degenerate spectrum is valid only by sequential change in τ (or k_o) over successive limited paths. In lieu of going directly from (34.23) to (34.24) by choosing $\tau = \tau_c$, one can expand $\psi(\hbar, z; \tau)$ of Eq. (34.21a) in a Taylor series about $\tau = \tau_c$. Then Eq. (34.21a) takes on the form

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$$p(\underset{\sim}{x}) = \frac{1}{4\pi}\left(\frac{2\pi i}{k_0|\hat{X}-\hat{S}|}\right)^{\frac{1}{2}} exp\left(ik_0|\hat{\underset{\sim}{X}}-\hat{S}|\right)\sum \frac{1}{l!}\left(\frac{|\hat{X}-\hat{S}|}{2k_0}\right)^l \psi^{(l)} I_l\left(ik_0 \hat{\underset{\sim}{X}}-\hat{S}\right)$$

$$\psi^{(l)} = i^l\left[\frac{\partial^2}{\partial z^2} - \left(\frac{\omega}{\bar{c}}\right)^2 - k_0^2\right]^l \psi\left(|\hat{X}-\hat{S}|, z\right)$$

$$I_l(\alpha) = \left(-\frac{\alpha}{2\pi}\right)^{1/2} e^{-\alpha}\cdot\int_0^\infty \frac{d\tau}{\tau}(\tau-1)exp\left[\frac{\alpha}{2}\left(\tau+\frac{1}{\tau}\right)\right]$$

(34.25)

Here, $\psi(\hat{X}-\hat{S}, z)$ is the solution of the usual parabolic equation (see Eq. 21.24) and is obtainable in the form of derivatives of $I_0(\alpha)$ through use of the recursion formulas of the Hankel functions. Eq. (34.25) is a version of the parabolic equation modified to correct for the stationery-phase error introduced by the range-depth dependency of the ocean sound speed C.

Returning to Eq. (34.22a) one observes that the key difficulty in attempting solution of this equation is just the same range depth dependency $C\left(z, \hat{S}+\mu\hat{R}\right)$. What is required is a formula for $\left(\frac{\omega}{\bar{c}(z)}\right)^2 + V\left(z; \frac{\tau'}{\tau}, \tau\right)$, identifiable as the square of the space dependent wave number. Noting that

$$k_{eff}^2 = \left(\frac{\omega}{\bar{c}(z)}\right)^2 + V\left(z;\frac{\tau'}{\tau},\tau\right) = \frac{\omega^2}{C^2\left(z, \hat{S}+\frac{\tau'}{\tau}(\hat{\underset{\sim}{X}}-\hat{S})\right)}$$

(34.26)

and using the known Fourier transform for V (Eq. 34.16) one arrives at an effective wavenumber defined by

$$k_{eff}^2(\underset{\sim}{x}) = \frac{-ik_0\bar{R}}{2\pi R(R-\hbar)}\int dx'\int dy'\left[\frac{\omega}{C(x',y',z)}\right]^2 exp\left(\frac{ik_0R[(x'-\hbar)^2+y'^2]}{2\hbar(R-\hbar)}\right)$$

(34.27)

in which R is the range, $\hbar$ is the path length between source and receiver, $k_0 = \frac{\omega}{\bar{c}}$, y' is the cross-range coordinate, and x' the range coordinate. This formula requires that at any range $\underset{\sim}{R}$ (and range coordinate $\hbar$) the effective wavenumber be calculated by integrating $\left(\omega/c(\underset{\sim}{x})\right)^2$ over the area of the <u>horizontal circle</u> (in the xy plane) at depth z. The diameter of the circle is effectively the greatest excursion of the scattering in the super-eikonal approximation. This point is discussed below.

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Cross Range Effects in the Superiekonal Approximation

We return to Fig. 34.1 and again consider two ray paths, namely B+C, and A=R starting and ending at common points. The angle of scattering in the supereikonal approximation is $\theta_s \overset{<}{\sim} \dfrac{\lambda}{L_c}$, where L_c is the characteristic size of the scatterer. The radius of the 1st Fresnel zone is $\sqrt{\lambda R}$. We choose a size scale by forming the ratio $F = L_c^2 / \lambda R$. Let y be the separation of two rays at some distance and construct a distance $\varrho(\hbar)$ such that at any range $\hbar$, one can set $\theta = \dfrac{y}{\varrho(\lambda)}$. Now if $(\varrho/R) \overset{>}{\sim} F$ then the supereikonal restriction on θ is satisfied and the separation $y \overset{<}{\sim} \lambda \varrho / L_c$. However if $(\varrho/R) < F$ then $y \overset{<}{\sim} \sqrt{\lambda \varrho}$. What is ϱ?. From the conditions of propagation of the two rays it is seen that $\varrho(0) = \varrho(R) = 0$. A simple function of range coordinate $\hbar$ that describes this condition is $\varrho(\hbar) = \hbar (R-\hbar)\mathcal{R}$. A plot of $\varrho(\hbar)$ vs. $\hbar$ from source to receiver at any range is a cigar-shaped figure. The maximum excursion of rays in this figure is restricted by the supereikonal requirement. This condition is paralleled by the elliptical figures of Tatarski described earlier in this section. Both delimit the cross-range influence of the medium inhomogeneities on the propagation of sound. The cigar-shaped figure gives the separation y at any $\hbar$. Hence the calculation of the k_{eff}^2 by areal integration over a circle described above is essentially done over a circle radius y.

Range of Validity of the Supereikonal Approximation

Callan and Zachariasen (Stanford Res. Inst. Tech. Report JSR-73-10 (1974)) have derived the following range limits to the validity of the supereikonal approximation. If L_H is the correlation length in the horizontal or range coordinate, $\delta c / c$ is the range dependent fluctuation in the ocean sound speed, and k_0 is the wavenumber associated with the average speed of sound (viz $k_0 = \dfrac{\omega}{c}$) then the maximum range $\mathcal{R}$ of valid application of the supereikonal is given by the inequality,

$$R < \max(\hbar_1, \hbar_2), \quad \hbar_1 = L_H \left(\frac{\delta c}{c}\right)^{-2/3}; \quad \hbar_2 = \frac{L_H}{(k_0 L_H)}\left(\frac{\delta c}{c}\right)^{-2}$$

In constrast the maximum range of the eikonal approximation is

$$R < \min(\hbar, \hbar_3), \quad \hbar_3 = L_H (k_0 L_H)^{4/3}$$

In the ocean L_H is some 1 to 10 km and $\delta c/c$ is of the order of 10^{-4}. At 100 Hz, $\omega = 200\pi$, $k_0 = 200\pi/1500$, $L_H \approx 10\,km$, thus

$$r_1 = \frac{10 \times 10^3}{(10^{-4})^{2/3}} \sim 4641\,km, \quad r_2 = 58.9\,km, \quad r_3 = 6.75 \times 10^8\,m$$

For this choice of ocean parameters and frequency the eikonal and supereikonal have the same range of validity (namely r_1).

35. Relation Between the Helmholtz Equation and the Parabolic Wave Equation

The small amplitude equation of steady state motion at frequency ω in a medium of sound speed $C(\underset{\sim}{x})$ is $\nabla^2 \psi(\underset{\sim}{x}) + \left(\omega^2/C^2(\underset{\sim}{x})\right)\psi(\underset{\sim}{x}) = 0$ (i.e. the Helmholtz equation). Here ψ is a field quantity (velocity potential or pressure). In a 2-dimensional waveguide with coordinates r, z this equation appears as $\psi_{rr} +$

$$(2a/r)\psi_r + \psi_{zz} + k_0^2 K(r,z)\psi = 0, \quad k_0 = \omega/c_0, \; K(r,z) = c_0/c(r,z)$$

and $a = 0$, 1/2 or 1 for Cartesian, cylindrical or spherical coordinates. The parabolic wave-equation in r, z coordinates is $2ik_0 p_r + p_{zz} + k_0^2\left[K(r,z) - 1\right]p = 0.$ Clearly p and ψ are related. It has been found by DeSanto (submitted to JASA) that the relation is an integral transform,

$$\psi(r,z) = A_0\, e^{A(r)} \int_0^\infty p\,(g(z,u))\, R(z;r,u)\, e^{B(r,u)}\, du \quad (35.1)$$

If one sets
$$A = (1-2a)\ln(r), \quad g(u) = u, \quad B(r,u) = (ik_0/2t)(r^2+t^2)$$
$$+ (a - \tfrac{3}{2})\ln(u)$$

then

$$\psi(r,z) = A_0\, r^{1-2a} \int_0^\infty p\,(r,u)\, R(z;r,u)\, e^{i\frac{k}{2u}(r^2+u^2)}$$
$$\times u^{a-\frac{3}{2}} du \quad (35.2)$$

This is a Fredholm integral equation of the first kind: ψ, p are presumed known and R is to be found. The ensuing analysis of the nature of $R\,(z;r,u)$ which is found by substituting this form ψ into the Helmholtz equation is found to be self-consistant. The nature of R depends on the nature of K. It is convenient to describe K as the sum of two quantities, $K(r,z) = K_1(z) + K_2(r,z)$, $K_2 \ll K_1$. Physically this is the assumption that there is a dominant dependence of the sound spped $K_1(z)$ on the depth coordinate z, and a small perturbation in sound speed K_2 dependent on depth and range r. Using this assumption in the substitution of

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the integral transform into the Helmholtz one arrives at the following differential equation to be satisfied by $\mathcal{R}$:

$$R_{\mu\hbar} + 2\left[(1-a)/\hbar + \frac{ik\hbar}{t}\right]R_{\hbar} + R_{zz} + 2ik_3 R_u$$

$$+ 2R_z \frac{\partial}{\partial z}\left\{\ell n\left[p(u,\bar{z})\right]\right\} = k_0^2\left[K_2(\bar{z},u) - K_2(\bar{z},\hbar)\right]\mathcal{R}$$

$$(35.3)$$

This form is difficult to solve in the general case. However it is observed that the condition $u = \hbar$ in Eq. 35.2 defines a <u>stationary phase point</u> of the integrand. It is then convenient to expand $\mathcal{R}(\bar{z}\,;\,u,\hbar)$ in a Taylor series about $\epsilon = u - \hbar$, i.e.

$$R(\bar{z}\,;\,u,\hbar) = 1 + \sum_m \epsilon^m \mathcal{R}^{(m)}(\hbar,\bar{z})$$

. Similarly the terms $\left[K_2(\bar{z},u) - K_2(\bar{z},\hbar)\right]$ and $\partial/\partial z\left\{\ell n\left[p(\bar{z},u)\right]\right\}$ are also expanded about ϵ . Substitution of these expansions into Eq. (35.3) yields a recursive system of equations in $\mathcal{R}^{(m)}$ (the m'th derivative of $\mathcal{R}$), each such equation belonging to a power of ϵ .

Thus the analysis of $\mathcal{R}$ by perturbation methods has led to a hierarchy of derivatives, $\mathcal{R}^{(0)}, \mathcal{R}^{(1)}, \mathcal{R}^{(2)}, \ldots,$ etc. Now it can easily be shown that if $K(\hbar,\bar{z})$ is independent of range $\hbar$, the first perturbation term $\mathcal{R}^{(0)}$ is a constant which can be taken to be unity. This condition $\mathcal{R} = \mathcal{R}^{(0)} = 1$ is equivalent to stating that the field point is exactly at the stationary point $u = \hbar$. $\mathcal{R}^{(0)}$ is identically the <u>range-independent solution</u>. The term $\mathcal{R}^{(1)}$, which corresponds to $\epsilon' = u - \hbar$, is arbitrary. For convenience $\mathcal{R}^{(1)}$ is set to zero. With the choice $\mathcal{R}^{(1)} = 0$, the terms $\mathcal{R}^{(2)} = 0$ and $\mathcal{R}^{(3)} = (k_0^2/6)\,\partial/\partial\hbar\left[K_2(\hbar,\bar{z})\right].$ Other $\mathcal{R}^{(m)}$ can similarly be obtained. Thus the perturbation series $\mathcal{R} = 1 + \sum \epsilon^m \mathcal{R}^{(m)}$ is constructed. Upon substitution into Eq. (35.2) one arrives at the following relation between ψ and p :

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$$\psi(r,z) = A_0 k_0^a \left(\frac{2\pi i}{k_0}\right)^{\frac{1}{2}} (k_0 r)^{-a} exp(ik_0 r) p(r,z)$$

$$\cdot \left\{ 1 + \left[\frac{ia}{k_0} - \frac{r^2}{2}\frac{\partial K_2}{\partial r}\right]\frac{1}{p}\frac{\partial p}{\partial r}\right.$$

$$\left. + \left(\frac{ir}{2R_0}\right)\frac{1}{p}\frac{\partial^2 p}{\partial r^2} - \left(\frac{3+a}{2}\right)r\frac{\partial K_2}{\partial r} + \cdots \right\}$$

$$\times \left\{ 1 + \mathcal{O}\left(\frac{1}{k_0 r}\right)\right\}$$

(35.4)

Several conclusions can be drawn from this result. First, since A_0 is arbitrary one can set $A_0 k_0^a \left(\frac{2\pi i}{k_0}\right)^{\frac{1}{2}} = 1$. If the resulting equation is limited to the first term on the r.h.s. one has $\psi(r,z) = (k_0 r)^{-a} exp(ik_0 r) p(r,z)$ This is precisely the usual parabolic approximation to ψ (see Sect. 16) which is thus seen to be the stationary phase approximation to the integral in Eq. (35.2). Second, the terms in $\partial p/\partial r$, $\partial^2 p/\partial r^2$, $\partial K_2/\partial z$ are seen to be corrections terms to the stationary phase solution. They allow the range dependence of $p(r,z)$ to be accounted for in respect to wave front curvature and changes of sound speed with range. Third, the correction terms can be numerically evaluated. A convenient procedure for

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doing this is to select a slab $\Delta \hbar$ units wide at range $\hbar_0$ and calculate the parabolics approximation $p(\hbar_0, \mp)$ from a marching algorithm with <u>known initial states.</u> Then $\Delta p / \Delta \hbar$ and $\Delta \left(\Delta p / \Delta r \right) / \Delta r$ are calculated by finite differences over the range of the slab thickness. Finally $\Delta K_2 / \Delta \hbar$ is numerically evaluated from known changes in K_2 over slab $\Delta \hbar$. Adding in all correction terms one finds ψ at $\hbar_0 + \Delta \hbar$, for any $\mp$. This is value of ψ is the initial state for the next slab.

<u>Conclusion:</u> In the preceeding pages the subject of a mathematical description of fluctuations of acoustic signals in the ocean has been reviewed in detail. The number and complexity of pertinent mathematical methods serves to emphasize the real difficulties of accommodating experimental results to match theoretical models. No single model currently exists which can predict fluctuations in acoustic signals due to fluctuations in the ocean background index of refraction, surface effects and moving platform for all frequencies of interest. Available models are further discussed in Part II of this series of reports, together with an extensive review of experimental data.

INDEX OF SYMBOLS

ν = kinematic viscosity, (m^2/sec)

η = coefficient of viscosity (N sec/m^2)

ε = dissipation rate (m^2/sec^3)

E = kinetic energy per unit wavenumber ($m^3 sec^2$)

T = kinetic energy per unit mass (m^2/sec^2); temp (K)

ϱ = density (N sec^2/m^4)

S = entropy (m^2/sec^2(K))

ζ = second coefficient of viscosity (N sec/m^2)

p = pressure (N/m^2)

σ_{ik}' = stress tensor (N/m^2)

β = coefficient of thermal expansion (K^{-1})

R = ideal gas constant (m^2/(K)sec^2).

C_p, C_v = specific heat at constant pressure, (constant volume) (m^2/sec^2,K)

= compressibility (m^2/N)

$\bar{U}_o$ = turbulent velocity (m/sec)

γ = normalized change in compressibility (non-dimensional)

γ_ϱ = normalized change in density (non-dimensional)

q = heat flux (N/m sec)

U_* = characteristic velocity (m/sec)

L = Monin-Obukov length (M)

x = constant (non-dimensional)

= matrix of momentum flux density (N/m^2)

u_i = acoustic particle velocity (m/sec)

= integral operator of scattering from inhomogeneities

= fluctuating temperature (K)

T_o = reference (or equilibrim) temperature (K)

ϱ' = fluctuating density (N sec^2/m^4)

ϱ_o = reference (equilibrium) density (N sec^2/m^4)

k = wave number (m^{-1})

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Π = dimensionless acoustic pressure (Eq. 3.7)

$\vec{V}$ = vector velocity (m/sec)

p_a = acoustic pressure (N/m^2)

C_σ = reference (or equilibrium) sound speed (m/sec)

u = rms turbulent velocity (reference state) (m/sec)

Λ_g = G. I. Taylor microscale for velocity (m)

Λ_θ = G. I. Taylor microscale for temperature (m)

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